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THEORY OF VIBRATIONS
FOR ENGINEERS

THE
THEORY OF VIBRATIONS FOR ENGINEERS

By

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EXTRACT FROM PREFACE TO FIRST EDITION

SINCE the introduction of high-speed machinery, and latterly the use of the Diesel engine for marine propulsion, the occurrence of vibration has been a source of grave anxiety, and a knowledge of the fundamental principles of the theory and the methods employed for eliminating the trouble is essential to the modern engineer. In many cases it is possible to predict the likelihood of serious vibrations being set up, and consequently to take steps for their prevention, during the design stage of the plant instead of by costly trial-and-error alterations after erection.

Valuable researches and contributions to the literature on the subject have been made in recent years, but since a high degree of mathematical attainment is necessary for their appreciation, they cannot receive the attention they deserve from the average engineer and student.

A section on the theory is included in most text-books on the Elasticity of Materials, but, of necessity, space will only permit a brief survey of the elementary principles, and the earnest reader is left unsatisfied.

A wide gap thus exists between the elementary treatment and the important analyses available, and the object of this volume is to bridge this gap and pave a way for higher study.

In method of presentation, the author has followed the lines which he has found successful after a considerable number of years' experience in teaching the subject to students, a number of whom had no knowledge of differential equations, but merely an elementary working knowledge of the calculus. His chief concern therefore has been to stress the physical meaning of the mathematical equations involved and to produce a visual picture of the motion resulting. For this purpose the vector method of solution for simple harmonic forced vibrations is of incalculable value—a method very familiar to alternating current electrical engineers, but which has been almost entirely neglected for the corresponding mechanical analogies.

Mathematics, however, has been in no sense avoided, but the essential purpose has been to discuss methods of practical utility and to give the reader a concrete foundation upon which to build.

Every endeavour has been made to acknowledge the sources of the information given and to indicate matter for further reading; and the author is specially indebted to H. M. Stationery Office for permission to make an extract from Reports and Memoranda Aeronautical Research Committee No. 1053; and also to the University of Liverpool for permission to use questions from recent examination papers as examples, a number of which have been fully worked out.

It is hoped that the book may prove useful to engineering students and all who may be making a first serious study of the subject.

PREFACE TO THE SECOND EDITION

WHEN the original text of this book was in the course of preparation in 1930, the author was informed that the Theory of Vibrations was of so little importance that a book for students would be of very limited appeal. To-day the story is very different and no engineer can afford to neglect the study of the subject.

The destruction of the whole of the first edition by a well-placed enemy bomb has given the author the opportunity of revising and enlarging the book, and, with such a wealth of interesting matter available, it has been difficult to decide what to omit in order to keep the text within the limits set.

The book has almost entirely been re-written and, with the passing of years, the author has found no reason to change his preference for employing vector methods of solution for certain differential equations, in fact he has been interested to note the increasing use of such illustrative methods.

The number of examples has been greatly increased, and as they have been devised chiefly for examination purposes their main object is to bring out a knowledge of principles with a minimum of arithmetical work.

In conclusion, the author would like to express his thanks to those readers who so kindly drew his attention to faults in the original text, and he would be grateful for similar help with the present edition should any errors be discovered.

E. B. COLE

The University
Liverpool, 1950

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INTRODUCTION

Units. Throughout this book the Engineers' Unit of Force, the *Pound-weight* (lb. wt.), will be used, and it has been thought advisable to recapitulate various units and definitions.

All physical quantities can be expressed in terms of Mass, Length and Time, and in the British system the corresponding fundamental absolute units are respectively the Pound (lb.), Foot (ft.), and Second (sec.).

Mass is the quantity of matter in a body, and is constant everywhere.

Weight is the *force* with which the earth attracts a body and varies over the earth's surface, but at any given place we may compare the masses of two bodies by comparing their weights.

The absolute unit of force is the *Poundal,* which is defined as the force which will give an acceleration of one foot per second per second to a mass of one pound.

An alternative way of measuring the magnitude of a force is to compare it with the force exerted by gravity upon a particular mass—in our case, one pound—and the force which gravity exerts on this one pound mass is termed a *Pound-weight.* It will be realized that this unit is not absolute but will vary with the locality.

Thus if g ft. per sec. per sec. is the acceleration due to gravity, a pound-weight is the force which will give this acceleration to a mass of one pound, i.e.

$$1 \text{ lb. wt.} = 1 \text{ lb. mass} \times g \text{ ft. per sec. per sec.}$$

If, therefore, a mass m lb. has an acceleration of f ft. per sec. per sec., the force applied is

$$m \times f \text{ poundals}$$

or

$$\frac{m \times f}{g} \text{ pound-weight.}$$

Again, we could define a pound-weight as the force which would produce an acceleration of one ft. per sec. per sec. in a mass of g lb., i.e.

$$1 \text{ lb. wt.} = g \text{ lb. mass} \times 1 \text{ ft. per sec. per sec.}$$

Engineers prefer to use the Pound-weight as their unit of force, the poundal being inconveniently small, and take g pounds as their unit of mass, so that

$$\text{Force in lb. wt.} = \frac{\text{mass in lb.}}{g} \times \text{acceleration in ft. per sec. per sec.}$$

It will thus be realized that, by our choice of units, the weight W of a body measured in lb. wt. is *numerically* equal to its mass m measured in lb. so that

$$\text{Force in lb. wt.} = \frac{m}{g} \cdot f = \frac{W}{g} \cdot f,$$

where $\frac{m}{g}$ or $\frac{W}{g}$ represents the *mass* of the body in engineers' units and will be denoted by the symbol M.

In dealing with systems having angular motion, Moments of Inertia, being analogous to mass in linear motion, will be measured in absolute units and will therefore be divided by g in the equations of motion so that torques will be measured in lb. wt.-ft. (usual abbreviation being lb. ft.).

In the solution of most of the examples g has been taken as 32 ft. per sec. per sec.

CHAPTER I

SIMPLE HARMONIC MOTION

WHEN a body moves to and fro so that every part of its motion recurs regularly it is said to move with *periodic motion.* There are, of course, many types of such motions, but by far the most important is that termed *Simple Harmonic,* which may be defined as " the motion of a body which is attracted to a fixed point in its path by a force which varies as the distance of the body from that point, measured along its path ". For example, let the path of the body be AOB and let O be the fixed point towards which the acceleration of the body is directed, i.e. towards which the force attracts the body (Fig. 1).

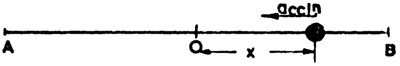

FIG. 1.

Let the body start from rest at A and, as it is being accelerated towards O, its velocity will increase, but, by hypothesis, its acceleration, being proportional to the distance from O, will decrease and become zero at O. The velocity at this point will be a maximum, since, as soon as the body passes O, the accelerating force, being always directed towards O, now retards the motion. The further the body moves from O the greater becomes this retardation, and consequently the less the velocity until the body comes to rest at some point B, when it immediately starts back again, since the attraction towards O is still acting. The point B will be situated at the same distance from O as A, since the retardation has been the exact reverse of the acceleration. Then by exactly the same reasoning as above it will be evident that the body will momentarily come to rest at the point A again, and, unless disturbed, will continue to undergo this cycle of operations indefinitely.

Let x be the distance of the body *from* O at any instant, then the force attracting the body *towards* O at this instant will be

$-Fx$, where F is the force at unit distance, the negative sign appearing since the force acts in the opposite direction to the displacement x.

Now if m is the mass of the body in lb., the force acting is also given by

$$\frac{m}{g}\frac{d^2x}{dt^2} \quad \text{or} \quad \frac{W}{g}\frac{d^2x}{dt^2} \text{ lb. wt.,}$$

being measured positive in the same direction as x.

Hence the equation of motion is

$$\underset{\text{Force accelerating body}}{\frac{W}{g}\frac{d^2x}{dt^2}} \quad = \quad \underset{\text{Force attracting body}}{-Fx}$$

or

$$\frac{W}{g}\frac{d^2x}{dt^2} \quad + \quad \underset{\text{Force \textit{overcoming} attraction}}{Fx} \quad = \quad 0.$$

Thus $\frac{d^2x}{dt^2}+n^2x=0$, where $n^2=\frac{Fg}{W}$ and is always positive.

The solution of this " simple harmonic " equation is

$$x=a\cos(nt+e),$$

where a and e are arbitrary constants.

[e.g. differentiating, $\frac{dx}{dt}=-an\sin(nt+e)$

$$\frac{d^2x}{dt^2}=-an^2\cos(nt+e)$$

$$=-n^2x].$$

Alternatively, since

$$\cos(nt+e)=\cos nt\cos e-\sin nt\sin e,$$

we may write

$$x=A\cos nt+B\sin nt$$

where $\left.\begin{aligned} A&=a\cos e\\ B&=-a\sin e\end{aligned}\right\}$ and are constants.

Thus $\tan e=-\frac{B}{A}$ and $a=\sqrt{A^2+B^2}$.

Physical interpretation of the solution. The motion is that of a point P, being the projection on any diameter of a point p which moves round the circumference of a circle of radius a, with uniform angular velocity n radians per sec., e.g. Fig. 2a:

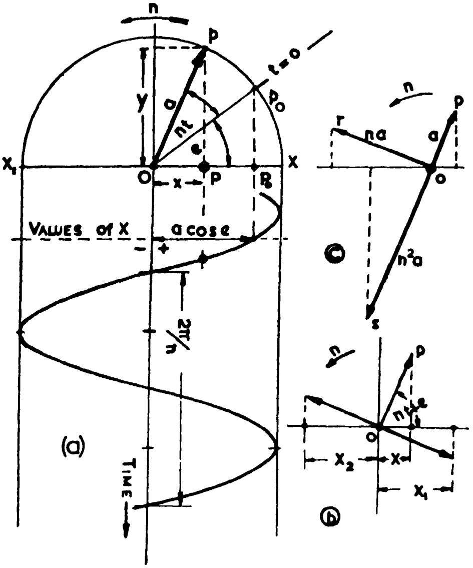

FIG. 2.

Let XX_1 be the diameter chosen, and let the point p be at p_0 at the time $t=0$, then after a time t the angle turned through will be $p_0Op=nt$.

If we denote the initial angle XOp_0 by e, the displacement x of P from the centre O is $x=a\cos(nt+e)$.

Hence P moves to and fro along XX_1 with S.H.M.

When $t=0$, the initial displacement x_0 is given by

$$OP_0=a\cos e=A,$$

while the initial velocity

$$\frac{dx_0}{dt}=-an\sin e,$$

so that

$$B=\frac{\text{initial velocity}}{n}.$$

Thus we see the meaning of the two constants a and e or A and B which come in since the body may start from any position with any velocity.

Consider again the equation $x = a \cos (nt + e)$, since $\cos (nt + e)$ varies between ± 1, the maximum displacement from the mean position is $\pm a$, hence a is called the *amplitude* of the motion.

The cycle recurs every time $nt = 2\pi$, 4π, etc., i.e. every time the point p makes a complete revolution, so that the time interval or *period* of the cycle is

$$T = \frac{2\pi}{n} \text{ seconds.}$$

Then $\frac{1}{T} = \frac{n}{2\pi}$ gives the number of complete cycles occurring in unit time, and is termed the *frequency*.

The term "cycles per second" is usually denoted by the abbreviation "cyps" or by the symbol $\sim$, and the angular velocity $n = (2\pi \times \text{frequency})$ is often called the *circular frequency*.

If we plot the values of x against time we obtain the cosine wave as shown in Fig 2*a*.

The radius $op = a$ will be called the radius arm or vector generating the S.H.M.

Another important term is the *phase* of the motion. Two bodies will be in the same phase if at any instant they have made exactly the same fraction of a complete cycle from corresponding positions, i.e. their generating vectors will have turned through exactly the same angle starting from corresponding positions. If we have one simple harmonic motion

$$x_1 = a_1 \cos (nt + e)$$

and another of the same period, but which in phase lags behind this by a quarter period, then this latter motion may be represented by $x_2 = a_2 \cos (nt + e - 90^\circ)$, the vector a_2 lagging behind the first by 90 degrees.

Thus the angle between the two vectors gives the difference in phase between the two motions. This angle will be constant if, as above, the two motions have the same period, and will vary if the periods differ, the two bodies coming in and going out of phase respectively as their generating vectors approach or recede.

Since $n^2 = \frac{Fg}{W}$, the period depends only on the magnitude of the load and the value of F, and does not depend in any way on the amplitude.

The amplitude depends on the initial displacement and the initial velocity imparted to the body.

Note that although we have taken the solution of the simple harmonic equation to be of cosine form, the motion could equally well have been represented by a sine wave

$$x_1 = a \sin (nt + e).$$

This would be generated by a vector of length a, *lagging behind* the cosine vector by 90 degrees, since with the usual trigonometrical conventions

$$x_1 = a \cos (nt + e - \pi/2) = a \sin (nt + e). \quad \text{Fig. 2 } (b)$$

A vector 90 degrees *ahead* of the cosine vector would produce a sine wave of opposite sign

$$x_2 = a \cos (nt + e + \pi/2) = -a \sin (nt + e). \quad \text{Fig. 2 } (b)$$

Again, if we projected the vector Op in Fig. 2*a* on the vertical OY instead of on OX the motion would be

$$y = a \sin (nt + e),$$

where the angle e is measured from OX as before.

Now the velocity of the motion, being $\frac{dx}{dt} = -an \sin (nt + e)$, can also be represented by the projection of a vector Or of length an and rotating with angular velocity n. This vector will be situated 90 degrees ahead of the displacement vector Op as shown in Fig. 2*c*, so that it will produce the negative sine wave. This wave reaches its maximum as the displacement wave passes through its mean, or in other words the body attains its maximum velocity as it passes through its mid position.

Similarly, the acceleration $\frac{d^2x}{dt^2} = -an^2 \cos (nt + e)$ can be represented by the projection of a vector Os of length an^2 and situated at 180 degrees to the displacement vector, thus producing the negative cosine wave, which gives the required acceleration towards the centre.

An important example of S.H.M. is the motion of the piston of a reciprocating engine, assuming an infinitely long connecting rod, since the piston will be at the same point on its stroke as the projection of the crankpin on the crank circle diameter which coincides with the stroke line.

If $R =$ the length of the crank ;
$\theta =$ the crank angle measured from the inner dead centre ;
$n = d\theta/dt =$ the angular velocity of the crank, assumed uniform ;

then, taking $t=0$ when the crank is at the inner dead centre, we have

Displacement of piston from its mid position

$$= R \cos \theta = R \cos nt.$$

$$\text{Velocity of piston} = -R \sin \theta \,.\, \frac{d\theta}{dt} = -Rn \sin nt.$$

Sum or difference of two simple harmonic motions of the same period but of different phases, along the same straight line. The result may be seen at once by writing the two motions in the form :

$$x_1 = A_1 \cos nt + B_1 \sin nt \,;$$
$$x_2 = A_2 \cos nt + B_2 \sin nt.$$

Then the sum

$$\begin{aligned}(x_1 + x_2) &= (A_1 + A_2) \cos nt + (B_1 + B_2) \sin nt \\ &= C \cos nt + D \sin nt,\end{aligned}$$

i.e. the resultant motion is simple harmonic of the same period but of different phase and amplitude from either component.

Alternatively, graphically by means of vectors, let the S.H.M.s be denoted by

$$x_1 = a_1 \cos (nt + e_1),$$
$$x_2 = a_2 \cos (nt + e_2),$$

then (Fig. 3*a*) let $Op_1 = a_1$, $Op_2 = a_2$ be the radius arms generating the motions of P_1 and P_2 of the above values along XX_1, i.e. at the instant shown:

$$OP_1 = x_1, \qquad OP_2 = x_2,$$

$$\angle P_1Op_1 = (nt + e_1), \qquad \angle P_2Op_2 = (nt + e_2).$$

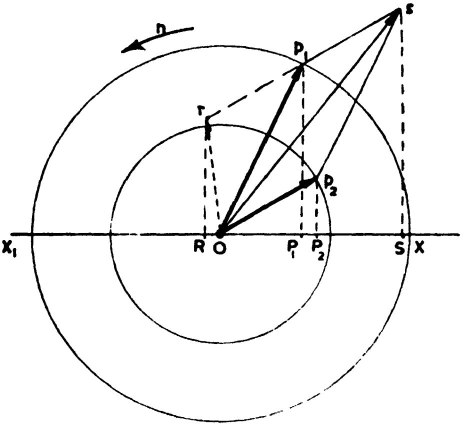

FIG. 3*a*.

Since the periods are the same, the radius arms rotate with the same angular velocity and in the same relative position to one another. These two radii are rotating vectors and may be added together in the usual manner, e.g. draw p_1s equal and parallel and in the same direction as Op_2, then Os is the vector sum of Op_1 and Op_2.

This vector Os revolves in the position shown relative to the component vectors and therefore with the same angular velocity, thus generating on XX_1 a simple harmonic motion of the same period as either component motion, and of amplitude $= Os$.

To prove that this S.H.M. is the resultant of the other two, we have

$$\text{Projection of } Os \text{ on } XX_1 = OS = OP_2 + P_2S.$$

But

$$P_2S = \text{projection of } p_2s$$

$$= \text{projection of } Op_1;$$

$$\therefore \; OS = OP_2 + OP_1 = x_2 + x_1.$$

In a similar manner we can obtain the corresponding result for the difference of the two motions, e.g. subtracting Op_2 from Op_1, draw p_1r equal and parallel but in the reverse direction to Op_2, then Or is the required vector difference and generates on XX_1 the resulting motion, its projection OR being easily seen to equal $P_2P_1 = x_1 - x_2$.

Consider a very important case of the above, when the two generating vectors are at 90 degrees (Fig. 3*b*), i.e.

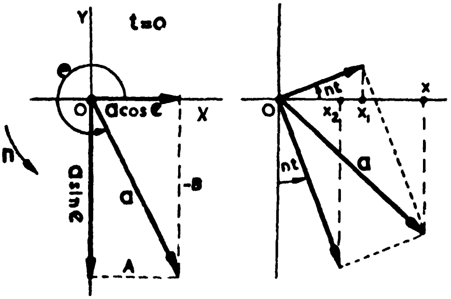

FIG. 3*b*

$$ox = ox_1 + ox_2,$$

where $$ox_1 = A \cos nt,$$

$$ox_2 = B \cos (nt - 90°) = B \sin nt.$$

When $t = 0$, the first vector of length A will lie in the projection line on the positive side, and the second of length B will lie along OY_1, since it lags by 90 degrees and in this position is about to generate the positive portion of a sine wave.

The resultant motion will therefore be generated by a vector of length $a = \sqrt{A^2 + B^2}$ having an initial phase angle e where

$$\tan e = -B/A,$$

and $$a \cos e = A,$$

$$a \sin e = -B,$$

so that $$x = A \cos nt + B \sin nt$$

$$= a \cos e \cos nt - a \sin e \sin nt$$

$$= a \cos (nt + e).$$

Thus we have shown vectorially the meaning of the two alternative forms of a S.H.M.

Note that with A and B both positive the resultant vector lies in the 4th quadrant.

It is important to be able to convert from one form of S.H.M. into the other and the signs of A and B indicate in which quadrant the vector is situated initially, e.g. in cosine form

A	B	e
+	+	4th
+	−	1st
−	−	2nd
−	+	3rd

$$\tan e = -B/A.$$

Suppose
$$A = +20,$$
$$B = +10,$$
$$\tan(-e) = B/A = \tfrac{1}{2}. \quad \text{[1st Quadrant]}$$
$$\therefore\ e = (360^\circ - 26^\circ\ 30'). \quad \text{[4th Quadrant]}$$

If in Fig. 3*c* angles are measured from OX as usual *but we now project on* OY, we have

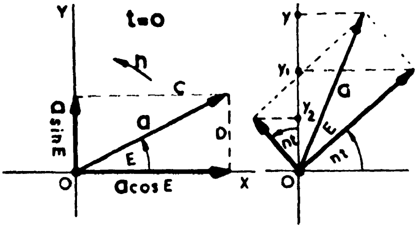

FIG. 3*c*.

$$\begin{aligned} oy &= a \sin(nt + E) \\ &= oy_1 + oy_2 \\ &= a \cos E \sin nt + a \sin E \cos nt \\ &= C \sin nt + D \cos nt, \end{aligned}$$

where
$$a \cos E = C,$$
$$a \sin E = D,$$
and
$$\tan E = \frac{D}{C}.$$

Sum of two simple harmonic motions, of different periods and phases, along the same straight line.

$$x = a_1 \cos (nt + e_1) + a_2 \cos (qt + e_2).$$

The two vectors Op_1 and Op_2 now rotate with different angular velocities n, q, and the length of the resultant vector Os will vary between $a_1 \pm a_2$ as the angle between the two primary vectors varies from 0 to 180 degrees.

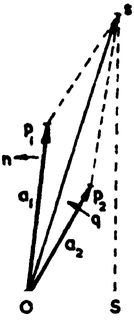

FIG. 4.

The resultant displacement OS, being given by the projection of the resultant vector, will depend entirely on the angular position of Os relative to the projection line as shown in Fig. 4, where Os is large but its projection OS is small, so that the maximum length of Os is not necessarily the maximum displacement. In order to gain a clear idea of the motion, two S.H.M.s, moving along *vertical* paths and having periods of 4 and 5 secs. respectively have been combined graphically in Fig. 5, both motions starting in phase from their mid-positions, i.e. the two generating vectors coincide initially.

It will be seen that the resulting motion is not simple harmonic, but it *is* periodic, and the *phase period* or time interval for the resultant wave to reproduce itself is represented by the length $AB = 20$ secs., during which each component motion has completed an exact number of periods, here 4 cycles of the slower motion (5 sec. period) or 5 cycles of the faster.

If the periods had been fractional, say $\frac{2}{3}$ sec. and $\frac{3}{5}$ sec., then

in 2 secs. the first motion would have completed 3 cycles and

in 3 secs. the second motion would have completed 5 cycles so that

in 6 secs., 9 and 10 cycles respectively would have been completed and the two motions would be in phase again.

Fig. 6 shows the resulting motion, 9 and 10 cycles of the component motions having been drawn on a length representing 6 seconds.

It will be noticed that the two periods chosen, and therefore the angular velocities *n* and *q*, are nearly, but not exactly equal.

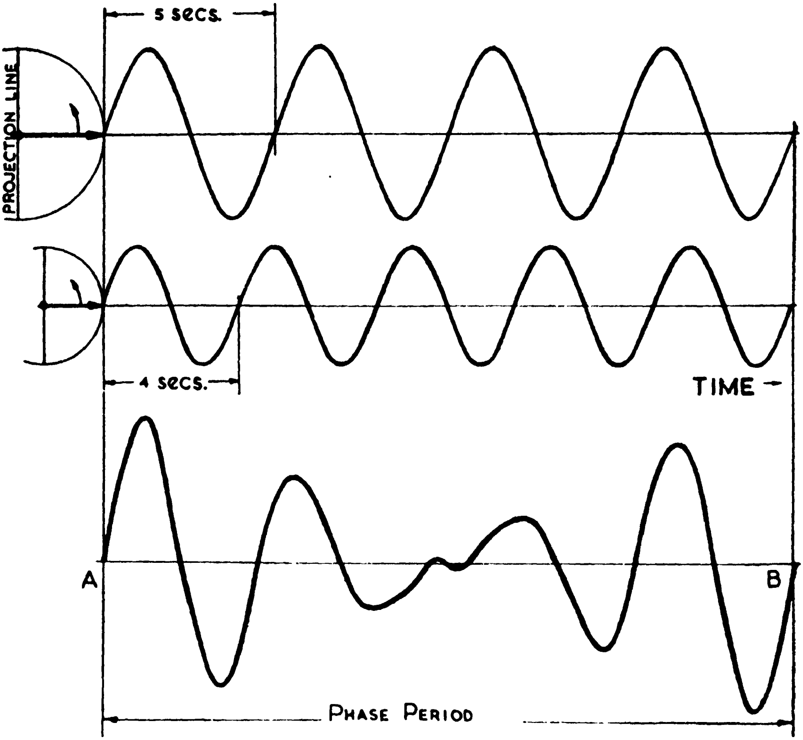

FIG. 5.

This is the most important case and is of particular interest. The vectors Op_1 and Op_2 are now rotating at approximately the same speed so that, *in one revolution*, their relative positions will hardly alter and their resultant *Os* will be very nearly constant during this time and will rotate with almost uniform velocity.

The motion is therefore approximately simple harmonic having an amplitude now equal to *Os*, which will vary very slowly from (a_1+a_2) to (a_1-a_2). The angular velocity of *Os*

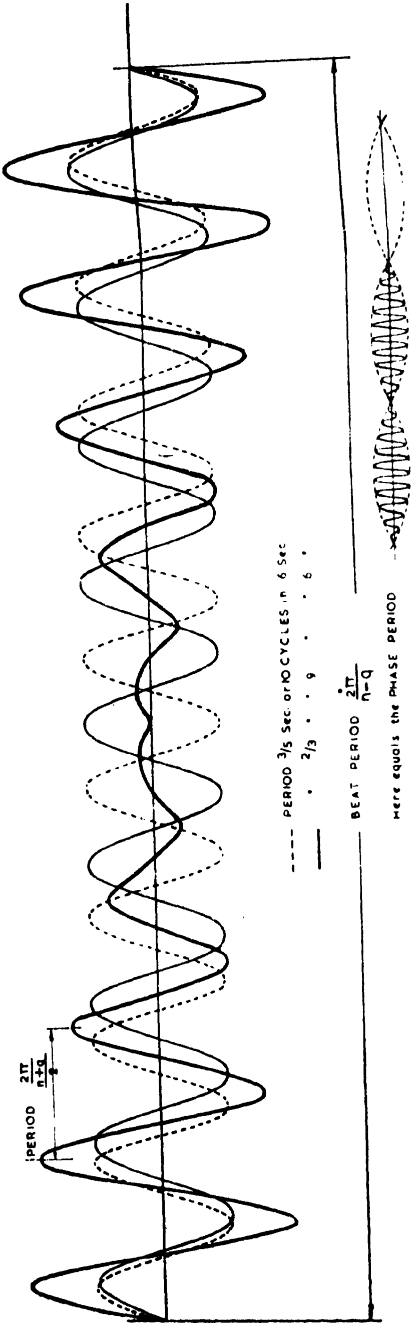
PERIOD $\frac{2\pi}{\frac{n+q}{2}}$
PERIOD 3/5 Sec. or 10 CYCLES in 6 Sec
2/3 . . 9 . . 6 .
BEAT PERIOD $\frac{2\pi}{n-q}$
Here equals the PHASE PERIOD

FIG. 6.

will be the mean of n and q so that the period will be $\dfrac{2\pi}{\left(\dfrac{n+q}{2}\right)}$.

The time interval between successive maxima or minima will be the time interval for one vector to rotate through one revolution *relatively* to the other, with relative angular velocity $n-q$, i.e. period $=\dfrac{2\pi}{n-q}$.

If the amplitudes of the two motions a_1, a_2 are both equal as in Fig. 6, then the resultant amplitude varies from $2a$ to zero, thus the effect is very pronounced and there are intervals of approximate rest.

Examining this case analytically, it is obvious that we must choose some initial positions for the vectors—in other words, we must assume some definite initial displacements and velocities for the primary motions.

Let us assume that at the start the two vectors, each equal to a, coincide, i.e. $e_1=e_2=e$, and let $e=0$, so that the motions start in phase from their maximum positions. (In Fig. 6 the two motions have been started from their mid positions.)

We thus have

$$x=a\cos(nt)+a\cos(qt)$$
$$=2a\cos\left(\frac{n-q}{2}\right)t\,.\,\cos\left(\frac{n+q}{2}\right)t.$$

When $q\to n$ this may be considered as

$$x=A\cos\left(\frac{n+q}{2}\right)t,$$

where
$$A=2a\cos\left(\frac{n-q}{2}\right)t,$$

which represents a simple harmonic motion of period $2\pi\Big/\dfrac{n+q}{2}$ with an amplitude A which will be approximately constant during this time, but which slowly fluctuates between 0 and $2a$.

The period of the cosine wave giving the value of A is $2\pi\Big/\dfrac{n-q}{2}$, but the time interval between successive maxima or minima of A will be half this value, i.e. $2\pi/n-q$ or intervals of 180 degrees of vector movement.

This rhythmic increase and diminution of amplitude is termed *beating*, the name being first used to describe this phenomenon as it occurs in music. Two simple tones of slightly different pitch (frequency) will produce an alternate increase and diminution in the volume (amplitude) of the sound, which becomes very pronounced as the two frequencies approach one another.

As another important illustration we have the tides of the sea which are the resultant effect of two simple harmonic motions, one produced by the attraction of the sun the other by the moon, and the superposition of the two causes a fluctuating amplitude, i.e. the spring and neap tides.

A most interesting practical use of sound beats is embodied in the Maihak Extensometer which is an instrument designed to measure the alteration in length of any structure in any given direction while under actual load.

The apparatus consists essentially of a steel wire which is stretched between grips at its ends over the portion of the structure under test. The grips are firmly fixed to the structure and thus the alteration in length of the wire will equal that of the structure in the same direction. This wire can be set into transverse vibration by means of an electromagnet on depressing a contact key, and thus a musical note will be emitted, the pitch of which depends on the tension in the wire, i.e. on how much the wire is stretched. The sound is conveyed to the ears electrically by means of headphones.

By rotating a dial the extension of an exactly similar calibrated wire can be altered at will, and when set into transverse vibration by depressing another key, the note it emits can be compared with that of the other wire.

To tune this calibrated wire to the wire on the test member whether for the initial setting under no load or when under load, each key is depressed alternately and the pitch of the notes produced roughly compared, the dial then being rotated in one direction or the other until the pitch of the notes appears about the same. Then, by depressing a third key, both wires can be caused to vibrate simultaneously, and, the notes being nearly of the same pitch, beats will be heard.

The dial is now rotated until the beats disappear, an operation that can be performed with great accuracy, the period of beating getting longer and longer as we approach synchronisation, until finally an even continuous note is heard.

The actual alteration in length of the member under test is therefore at once obtained from the initial and final dial settings, the calibrated wire having suffered the same deformation, or if it has been found more convenient to use a tuning wire of different dimensions it is a simple matter to correct for this.

Combination of two simple harmonic motions of the same period along two lines at right angles. Let the two motions be

$$x = a_1 \cos(nt + e_1),$$
$$y = a_2 \cos(nt + e_2).$$

This is best examined graphically (Fig. 7). Let OX, OY be the two lines at right angles. Let $OA = a_1$ and $\angle XOA = e_1$, so

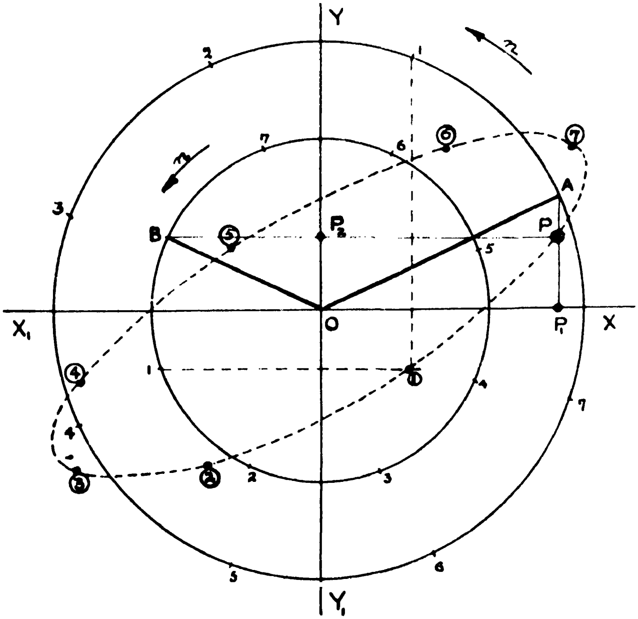

FIG. 7.

that OA by its rotation with angular velocity n will generate on XX_1 the S.H.M. $a_1 \cos(nt + e_1)$. Similarly, let $OB = a_2$ and $\angle YOB = e_2$, so that OB revolving with angular velocity n generates on YY_1 the S.H.M. $a_2 \cos(nt + e_2)$.

Initially then, we are at P_1 for the S.H.M. along XX_1 and at P_2 for that along YY_1, hence if these two displacements are given to a point simultaneously, this point will therefore be at P.

Similarly, by dividing each circle into the same number of equal intervals starting from A and B respectively, and joining up corresponding projection lines on XX_1 and YY_1 we may plot the position of the point P at these intervals. It will be seen that the locus of P is an *ellipse.*

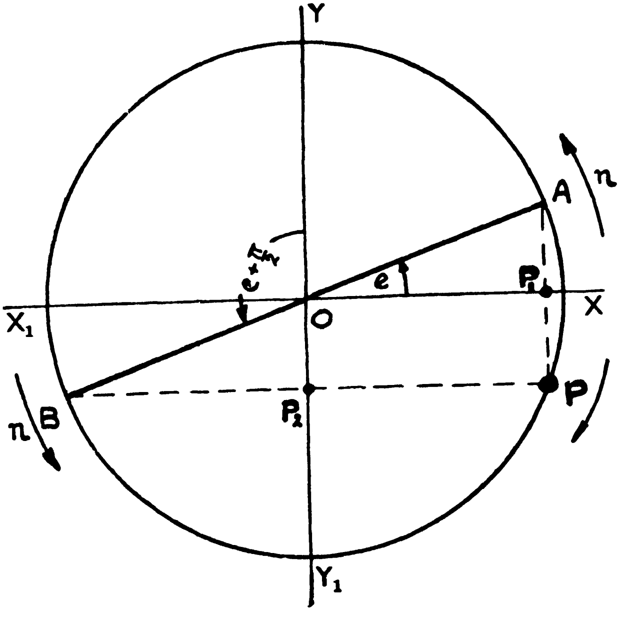

FIG. 8*a*.

Two special cases arise :

(1) If the amplitudes a_1, a_2 are each equal to a, but the phases differ by 90°, then the locus of P is seen to be a circle of radius a (Fig. 8*a*). *Uniform circular motion can therefore be considered as made up of two simple harmonic motions along perpendicular lines, having equal periods and amplitudes, but phases differing by* $\frac{1}{4}$ *period.*

Note that the angles have been measured from OX and OY respectively, and the point P travels round the circle in the *opposite* direction to the vectors. If we had measured the angle of the second motion from OY_1, then OB would coincide with OA, and hence P with A. P now rotates in the same direction as the vectors, and the projection OP_2 of OB on YY_1 is then $a \sin (nt+e)$, where this angle is measured from OX.

This conception is of the utmost importance and will be referred to again.

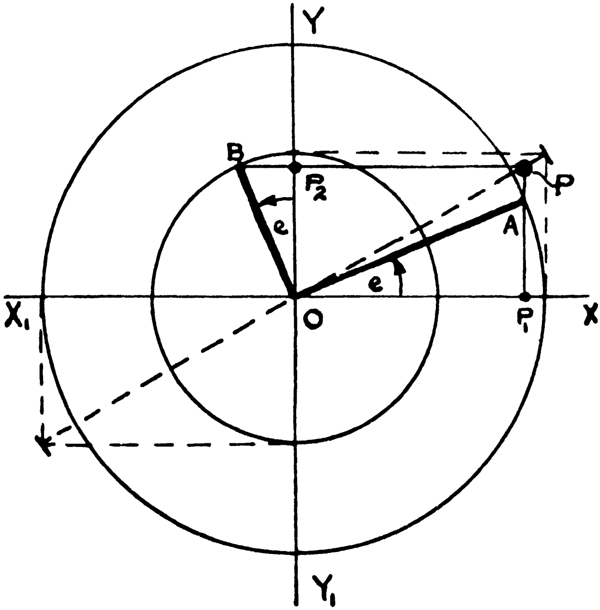

FIG. 8b.

(2) If the simple harmonic motions are of different amplitude, but of the same phase, then the resulting motion is simple harmonic of the same period along a straight line of slope a_2/a_1 with the axes OX and OY, and amplitude $\sqrt{a_2^2+a_1^2}$ (Fig. 8b).

CHAPTER II

FREE OR NATURAL VIBRATIONS OF SYSTEMS HAVING ONE DEGREE OF FREEDOM.

Longitudinal oscillations of a single concentrated load supported by an elastic structure, the mass of which will be assumed to be negligible in comparison with that of the load. By an elastic structure is meant one that obeys Hooke's Law, i.e. the extension of the elastic system is proportional to the load applied. No materials in engineering use are perfectly elastic but most may be considered so within certain limits. Hence all our investigations will be conducted on the assumption that we are working "within the limits of elasticity".

Thus if F is the force required to produce unit deflection of the structure, i.e. the *stiffness* of the structure, Fx = the force required for an extension x.

To take a definite example, consider a load of mass m supported by a vertical spring (Fig. 9).

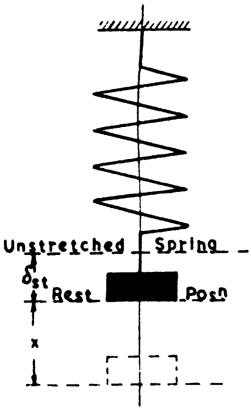

FIG. 9.

When the load is first attached to the unstretched spring its weight W will cause an extension δ_{st} where

$$W = F\delta_{st} \quad \text{or} \quad \delta_{st} = \frac{W}{F}.$$

The load will remain in equilibrium in this position, being balanced by the tension in the spring, the displacement being termed the " static displacement ".

If we pull the load down a distance x from its rest position the force required will be Fx, and the spring will exert a similar force upwards, which, upon releasing the mass, will accelerate it towards its rest position, the equation of motion therefore being

$$\frac{W}{g}\frac{d^2x}{dt^2} = -Fx,$$

or

$$\frac{d^2x}{dt^2} = -\frac{Fg}{W}x = -n^2x.$$

The negative sign appears since x is measured from the rest position while the acceleration is towards that position.

This is the equation for simple harmonic motion as deduced in Chapter I.

Thus the mass will oscillate about its equilibrium position with simple harmonic motion, the period of which is

$$\frac{2\pi}{n} = 2\pi\sqrt{\frac{W}{Fg}}$$

$$= 2\pi\sqrt{\frac{\text{Mass}}{\text{Force to produce unit extension of the elastic structure}}}$$

also since $\frac{W}{F} = \delta_{st}$,

period $$= 2\pi\sqrt{\frac{\delta_{st}}{g}}.$$

These oscillations are termed *free* or *natural* oscillations, being set up by an impulse or the sudden addition or removal of a portion of the load. If the starting impulse is not again repeated, these oscillations will continue until damped out by friction.

In our equation of motion we have so far neglected the effect of friction, and for many cases that arise in practice, e.g. systems vibrating in air, this is true enough ; but friction, however small, is always present, and *in time* these oscillations will be damped out.

We shall deal later with systems subjected to friction.

Besides external friction, the energy of vibration may be destroyed by internal friction. No substance is truly elastic, and the continual bending backwards and forwards will cause friction between its molecules, thus dissipating the energy in heat.

In the above simple case we have assumed that the load can vibrate in a longitudinal direction only, i.e. we have introduced certain constraints which prevent any other motion, angular or transverse, which could otherwise occur. The system thus possesses *one degree of freedom* only, since its configuration at any instant can be expressed in terms of a single variable or co-ordinate, which in this case is the displacement of the load from its rest position, *and the period of vibration of any such system can always be found by determining the stiffness of the elastic supports in the direction in which the motion is allowed to take place,* i.e. the force F, which, applied at the point of attachment of the load, will produce unit deflection of the structure—or, what amounts to the same thing, by determining the static deflection under the load W.

Let us see how this may be applied to a single load supported by an elastic rod. We shall, as above, assume that longitudinal oscillations only are permitted.

Longitudinal oscillations of a single load supported by an elastic rod, the mass of which is negligible in comparison with that of the load (Fig. 10).

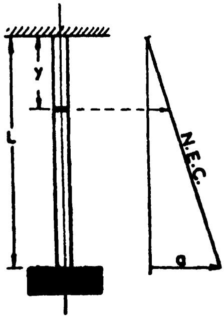

FIG. 10.

Let L = the length of the rod *when unstrained*, i.e. before the load is applied.

A = the area of cross-section of the rod.

E = the modulus of elasticity.

F, as will always be used, = the force to produce unit deflexion of the elastic structure at the point of application of the load.

Then, with exactly similar reasoning as for a loaded spring, we obtain

$$\text{Period} = 2\pi\sqrt{\frac{W}{Fg}}.$$

To determine the stiffness F, we have

$$\text{Stress} = E \,.\, \text{strain},$$

i.e.

$$\frac{\text{Force}}{A} = E \,.\, \frac{\text{extension}}{L}.$$

When the extension is unity, the force required is F,

so that

$$\frac{F}{A} = E \,.\, \frac{1}{L},$$

or

$$F = \frac{A \,.\, E}{L};$$

$$\therefore \text{ Period} = 2\pi\sqrt{\frac{WL}{AEg}},$$

or

$$\text{Frequency} = \frac{1}{2\pi}\sqrt{\frac{AEg}{WL}}.$$

Note that the equilibrium position of the load does not in any way affect the period of the vibration, which takes place about this position as mean. If the load is suspended vertically the elastic support is extended by an amount δ_{st} when the load is at rest. If, however, the same system is arranged to vibrate in a horizontal plane, the rod or other structure will be unstrained when the load is at rest.

Simple harmonic angular oscillations. The simple pendulum. This consists of a heavy particle (of mass m and weight W) which is suspended from a fixed point by a light inextensible cord.

In Fig. 11, let O be the fixed point, about which the mass can swing in a circular arc, making oscillations in one plane about the vertical rest position OA. Let the mass be displaced a horizontal distance x from the vertical OA, the cord making an angle θ with this line.

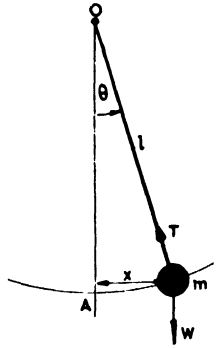

FIG. 11.

Then a torque $-Wx$ acts on the system tending to restore it to its equilibrium position, and when the mass is released the equation of motion will be

$$\frac{I}{g}\frac{d^2\theta}{dt^2} = -Wx = -Wl \sin\theta,$$

where $I = ml^2 = Wl^2 =$ the moment of inertia of the particle about O in absolute units, so that I/g is the value in engineer's units;

$l =$ the length of the cord;

the negative sign, as before, appearing since the displacement is measured away from the rest position, while the torque acts towards this position.

Now, *provided that this displacement is small*, $\sin\theta$ is nearly equal to θ radians, and we may therefore write

$$\frac{Wl^2}{g} \cdot \frac{d^2\theta}{dt^2} = -Wl\theta,$$

or

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l} \cdot \theta = -n^2\theta,$$

where $n^2 = \frac{g}{l}$.

Thus, the motion is simple harmonic, the period being

$$\frac{2\pi}{n} = 2\pi\sqrt{\frac{l}{g}}.$$

It will be interesting to obtain this result from a slightly different point of view:

Let $x =$ the horizontal displacement of m from the vertical rest position. The vertical displacement of m is then

$$l - l\cos\theta = l(1 - \cos\theta) = l \, . \, 2\sin^2\frac{\theta}{2}.$$

If θ is small, this movement is negligible, being a small quantity of the second order.

Assuming no vertical movement of the particle, there will be no force causing acceleration in this direction, so that the vertical component of the tension T in the cord will be equal to the weight W, i.e.

$$T\cos\theta = W,$$

and, since θ is small, for all practical purposes we may assume that

$$T = W.$$

Thus, when the particle is displaced a horizontal distance x

from its rest position, the restoring force acting is the horizontal component of the tension T, and the equation of motion is

$$\frac{W}{g}\frac{d^2x}{dt^2} = -T\sin\theta = -W\cdot\frac{x}{l};$$

$$\therefore\ \frac{d^2x}{dt^2} = -\frac{g}{l}\cdot x,$$

which represents a simple harmonic motion of period

$$2\pi\sqrt{\frac{l}{g}}.$$

The compound pendulum. This consists of any rigid body capable of making angular oscillations about a horizontal axis.

In Fig. 12, let G be the centre of gravity of the body and O the point of suspension.

Then, as for a simple pendulum, for a small angular displacement θ from the vertical rest position OA, the restoring torque is very nearly equal to $-W.OG.\theta$; and when the body is released the equation of motion will be

$$\frac{I_0}{g}\cdot\frac{d^2\theta}{dt^2} = -W.OG.\theta,$$

where I_0 = the moment of inertia of body about the axis of oscillation through O,

$= Wk_0^2$,

where k_0 is the radius of gyration about O.

$$\therefore\ \frac{d^2\theta}{dt^2} = -\frac{OG.g}{k_0^2}\cdot\theta$$

$$= -n^2\theta$$

where $n^2 = \dfrac{OG.g}{k_0^2}$.

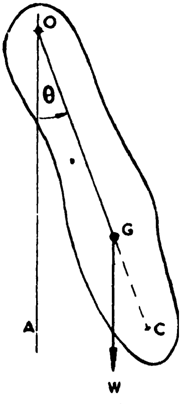

FIG. 12.

Hence the motion is simple harmonic, the period being

$$\frac{2\pi}{n} = 2\pi\sqrt{\frac{k_0^2}{OG.g}}.$$

The length L of a simple pendulum which would give the same period, i.e. the length of the *equivalent simple pendulum,* is

$$L=\frac{k_0^2}{OG}.$$

Now if k_G = the radius of gyration of the body about an axis through the centre of gravity G parallel to the axis of oscillation, then

$$k_0^2=k_G^2+OG^2.$$

Thus the period may be written as

$$2\pi\sqrt{\frac{k_G^2+OG^2}{OG\,.\,g}},$$

and
$$L=\frac{k_G^2+OG^2}{OG}=\frac{k_G^2}{OG}+OG.$$

Considering again Fig. 12, produce OG to C such that $OC=L$, the length of the equivalent simple pendulum.

The point O about which the body has been oscillating is termed the *centre of suspension,* but suppose we now put the pivot at C and allow O to swing below C, then the period will be

$$2\pi\sqrt{\frac{k_G^2+GC^2}{GC\,.\,g}},$$

or, the length of the equivalent simple pendulum is now

$$L'=\frac{k_G^2+GC^2}{GC}=\frac{k_G^2}{GC}+GC,$$

But
$$L=\frac{k_G^2}{OG}+OG,$$

and, by construction, $L=OC=OG+GC$,

so that
$$GC=\frac{k_G^2}{OG}\quad\text{or}\quad OG=\frac{k_G^2}{GC},$$

which, if substituted in the expression for L', gives

$$L'=OG+GC=OC=L.$$

The point C is called the *centre of oscillation,* and the result shows that the simple equivalent pendulum, and therefore the periodic time, is the same whether the body swings about O or C, that is, *the centres of suspension and oscillation are interchangeable.*

It should be noticed that since $k_G^2 = GO \cdot GC$, the lengths GO, GC are only equal when each is equal to k_G, in which case the centre of gravity is equidistant from O and C.

Angular oscillations controlled by the torsion of an elastic rod. In the following sections we shall study the angular vibrations of different shaft systems carrying concentrated loads, which are of uniform character such as pulleys, flywheels, etc., and the moments of inertia of which are large compared with that of the shafting. The shafting will therefore be assumed to possess no inertia, having elastic properties only.

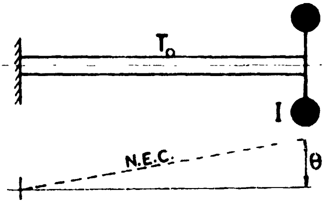

FIG. 13.

Case (1). *Torsional oscillations of a flywheel, or similar load, at the end of a uniform shaft fixed at the other end* (Fig. 13).

Let L = the length of the shaft.

J = the second moment of area of cross-section, which for a solid circular shaft is $\frac{\pi r^4}{2}$, r being the radius.

C = the modulus of rigidity.

I = the moment of inertia of the flywheel about the axis of the shaft, in *absolute units.*

Now from books on the " Strength of Materials " we know that if a torque T is applied at the free end of such a shaft, the angle of twist θ (radians) at this end is given by

$$\frac{T}{J} = C \cdot \frac{\theta}{L},$$

or

$$\text{Torque applied} = \frac{CJ}{L} \cdot \theta = T_0 \theta,$$

where $T_0 = \frac{CJ}{L}$ = the torque to produce unit angle of twist, i.e. the stiffness of the shaft in torsion.

Thus, if the flywheel is turned through an angle θ and released, the elastic restoring torque is $-T_0\theta$ and the equation of motion is

$$\frac{I}{g}\frac{d^2\theta}{dt^2}=-T_0\theta,$$

or

$$\frac{d^2\theta}{dt^2}=-\frac{T_0g}{I}\cdot\theta=-n^2\theta,$$

where $n^2=\frac{T_0g}{I}$.

Thus the motion is simple harmonic, the frequency being

$$\frac{n}{2\pi}=\frac{1}{2\pi}\sqrt{\frac{T_0g}{I}}=\frac{1}{2\pi}\sqrt{\frac{CJg}{IL}}.$$

Note that this is analogous to the result obtained for the longitudinal oscillation of a load on a rod, moment of inertia being substituted for mass, second moment of area for area, and modulus of rigidity for modulus of elasticity.

The following cases of torsional oscillation of shafts carrying concentrated loads are important. Although we shall consider the shafts in torsional oscillation, analogous results may easily be deduced for similar systems undergoing longitudinal oscillations by substituting W for I, E for C and A for J as noted above.

It will be remembered that in order to obtain the frequency of the longitudinal oscillations of a single load on a rod or any elastic structure all that we had to determine was the value of the stiffness F; here similarly, in order to obtain the frequency of the torsional oscillations of a single load, we have merely to determine the value of T_0 the torque to produce unit angular displacement of the elastic support at the point of application of the load, i.e. the torsional stiffness of the structure at the load point.

Case (2). *A single flywheel at the free end of a series of shafts* (Fig. 14).

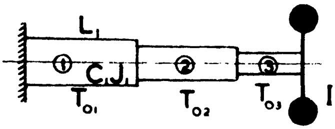

FIG. 14.

Let I = the moment of inertia of the flywheel (absolute units) and let the quantities referring to each shaft be denoted by the suffixes 1, 2, ... etc., respectively, so that the torsional stiffness of each section will be respectively

$$T_{01} = \frac{C_1 J_1}{L_1}; \quad T_{02} = \frac{C_2 J_2}{L_2}, \text{ etc.}$$

Then if we apply a torque T_0 at the load so as to give it an angular deflexion of unity, this torque will be transmitted through to each shaft, hence

Shaft (1) will be twisted through an angle $\theta_1 = \frac{T_0}{T_{01}}$ radians,

Shaft (2) will be twisted through an angle $\theta_2 = \frac{T_0}{T_{02}}$ radians, etc.,

and the sum of these angles of twist gives the total angular deflexion at the flywheel, which was given as unity, i.e.

$$\theta_1 + \theta_2 + \theta_3 + \ldots = 1,$$

or
$$\frac{1}{T_{01}} + \frac{1}{T_{02}} + \frac{1}{T_{03}} + \ldots = \frac{1}{T_0}.$$

This expression enables us to determine the *equivalent stiffness* T_0 of a series of shafts as shown.

The frequency is therefore

$$\frac{1}{2\pi}\sqrt{\frac{T_0 g}{I}} \quad \text{as before.}$$

Case (3). *A single flywheel carried at the junction of two shafts whose other ends are fixed* (Fig. 15).

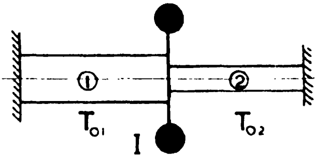

FIG. 15.

Here a torque T_0 applied at the load point will cause each shaft to twist through unit amount, thus the equivalent torsional stiffness T_0 of the whole system is equal to the sum of the two stiffnesses or

$$T_0 = T_{01} + T_{02}.$$

Case (4). *Two flywheels, one at each end of a rotating shaft* (Fig. 16).

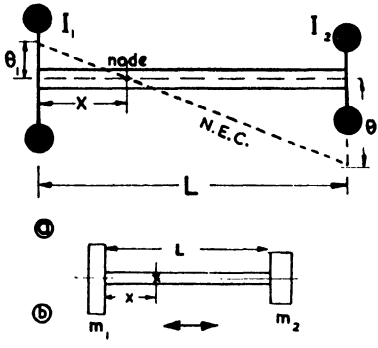

FIG. 16.

In this chapter we have been discussing the vibration of systems having one degree of freedom and constrained to perform one type of motion only.

For any system carrying x concentrated loads and having its elastic support fixed at one or both ends, the number of degrees of freedom, when a given type of vibration only is permitted, is equal to the number of loads, since the configuration at any instant can be denoted in terms of the displacements of the loads *relative to their fixed rest position.*

If, however, the system has no fixing, as in this case, the number of coordinates required to specify the configuration at any instant, say in torsional oscillation, will be $x - 1$ since the displacements must be stated *relative to one of the loads.*

This system thus has only one degree of freedom for the specified type of oscillation, here torsional.

The shaft when not in oscillation is assumed to rotate with uniform speed, and any torsional oscillations that may be set up will be superimposed on this uniform motion, so that we can consider matters as if the shaft was at rest.

When the system is set into torsional oscillation the flywheels will be moving in opposite directions, the shaft ends therefore being twisted in opposite directions, so that at some point X along the shaft, distant (say) x from I_1, there will be a stationary point or *node,* as it is called.

Since every point of the system is assumed to be vibrating with S.H.M. of the same frequency, the frequency of the whole is equal to that of any part. Here the node, being a stationary point, divides the system into two parts, each of which is similar to Case (1) above.

Thus the frequency of the whole system is equal to that of I_1 on a length of shaft x up to node, or I_2 on the length $(L-x)$, i.e.

$$\text{Frequency} = \frac{1}{2\pi}\sqrt{\frac{T_{01}g}{I_1}} = \frac{1}{2\pi}\sqrt{\frac{T_{02}g}{I_2}};$$

$$= \frac{1}{2\pi}\sqrt{\frac{CJg}{I_1 x}} = \frac{1}{2\pi}\sqrt{\frac{CJg}{I_2(L-x)}};$$

$$\therefore\ I_1 x = I_2(L-x);$$

$$\therefore\ \frac{x}{L} = \frac{I_2}{I_1+I_2},$$

which gives the position of the node X, and it is seen to divide the shaft into two lengths in the inverse ratio of the moments of inertia of the loads.

The frequency is at once obtained by substituting the value of x or $(L-x)$ in its respective expression above.

Alternatively, working in terms of stiffness, if T_0 is the torsional stiffness of whole shaft length L, then

$$T_{01} = T_0 \cdot \frac{L}{x}$$

(shorter length, therefore proportionally stiffer),

$$T_{02} = T_0 \cdot \frac{L}{L-x}.$$

Instead of the above system, consider the corresponding one in longitudinal vibration, i.e. two masses m_1, m_2 coupled by a rod or spring and free to vibrate longitudinally in some sort of frictionless guide (Fig. 16*b*).

The length of the elastic connexion now varies during vibration but the node will form a stationary point in space.

To solve this problem by the method of nodes, let $x=$ the distance of the node from one mass at the instant that L is the length of the elastic connexion. The procedure is exactly the same as before except that an equation in x/L is formed.

Case (5). *Two loads at the ends of a shaft free to revolve, the shaft having sections of different diameter* (Fig. 17).

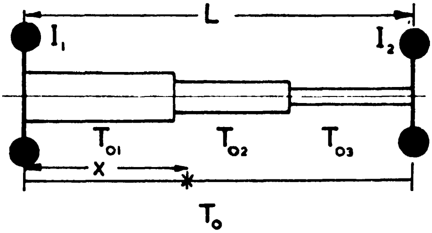

FIG. 17.

As for the previous case, a node will occur somewhere along the shaft.

The simplest procedure will be to obtain the equivalent stiffness T_0 of a uniform shaft which could replace the actual, as was done for Case (2) above :

$$\frac{1}{T_0}=\frac{1}{T_{01}}+\frac{1}{T_{02}}+\dots .$$

The system then reduces to Case (4), and finding the position of the node for this equivalent system enables the frequency to be calculated.

If we wish to determine the real position of the node on the actual shaft, the argument would be as follows :

Suppose the distance of the node from I_1 in the equivalent system is x, then the stiffness of this length $T_{0x}=T_0 \cdot \frac{L}{x}$. If this stiffness is *less* than T_{01} of section (1), a longer length of the actual shaft is necessary to lower its stiffness, i.e. the node is beyond section (1), say at a distance y into section (2).

The stiffness T_{0y} of this length $y=T_{02} \cdot \frac{L_2}{y}$;

$$\therefore \frac{1}{T_0 \cdot \frac{L}{x}}=\frac{1}{T_{01}}+\frac{1}{T_{02} \cdot \frac{L_2}{y}},$$

whence y/L_2 can be determined.

If y happened to be greater than L_2, the node is in section (3), but it would obviously be easier to check this first by working from the other end.

As an example, taking simple numbers, in their correct units, let

$$I_1 = 3I_2\,; \quad \therefore \quad x = \tfrac{1}{4}L.$$

$$\left.\begin{array}{l} T_{01} = 3 \\ T_{02} = 2 \\ T_{03} = 1 \end{array}\right\} \quad \begin{array}{l} \therefore \dfrac{1}{T_0} = \dfrac{1}{3} + \dfrac{1}{2} + 1 = \dfrac{11}{6}, \\ \text{or} \quad T_0 = \dfrac{6}{11}. \end{array}$$

∴ Stiffness of length x up to node in equivalent system

$$= 4T_0 = 2\tfrac{2}{11}:$$

∴ Stiffness of remaining length $L - x$ in equivalent system

$$= \tfrac{4}{3}T_0 = \tfrac{8}{11}.$$

These values being less than T_{01} and T_{03} respectively show that actual node is in section (2).

Thus
$$\frac{1}{24/11} = \frac{1}{3} + \frac{1}{\dfrac{2L_2}{y}};$$

$$\therefore \frac{L_2}{y} = 4 \quad \text{or} \quad \frac{y}{L_2} = \frac{1}{4},$$

i.e. the node is situated $\frac{1}{4}$ the way along section (2).

Equivalent length. In a multiple system comprising connecting shafts of different diameters and perhaps of different materials, instead of finding the equivalent stiffness of a uniform shaft to replace the actual, it sometimes simplifies matters if each section is reduced to an *equivalent length* of shaft of some convenient diameter and material, the length being such that the elastic conditions are unaltered, i.e. T_0 is unchanged.

Thus
$$T_0 = \underset{\text{(actual)}}{\frac{C_A J_A}{L_A}} = \underset{\text{(equivalent)}}{\frac{C_E J_E}{L_E}}.$$

For solid circular shafts, $J = \dfrac{\pi D^4}{32}$;

$$\therefore L_E = L_A \cdot \frac{C_E}{C_A} \cdot \frac{D_E{}^4}{D_A{}^4}.$$

For example, 1 foot of 6-inch diameter shaft is equivalent to $1 \times (\frac{12}{6})^4 = 16$ feet of 12-inch diameter shaft of the same material.

Normal elastic curve. This most important curve gives the displacements of every point of the system at any instant *during free vibration* relative to that of a convenient load, this displacement therefore being taken as unity. If the connexion between two loads, or between a load and its fixing, is assumed to be perfectly elastic but massless, then during vibration the displacement of any point relative to one end of the connexion considered will be proportional to the distance from that end—in other words, the normal elastic curve (N.E.C.) is made up of straight lines.

The N.E.C.s for a single load in longitudinal and in torsional oscillation on an elastic rod or shaft are shown in Figs. 10 and 13, the displacement of a point (Fig. 10) distant y from the fixed end being $a \cdot \frac{y}{L}$ where a is the displacement of the load at the free end at the same instant. Fig. 16 shows the N.E.C. for two loads in torsional oscillation. N.E.C.s for multiple systems are shown in Figs. 20, 25.

The energy equation. Consider the system of a mass of weight W in free vibration at the end of a vertical elastic rod—the mass of the rod, as before, being neglected. The system will possess both kinetic and potential energies, and, as we are assuming no damping forces, from the law of conservation of energy we have

Kinetic Energy + Potential Energy = Constant.

The Potential Energy consists of two parts :

(1) that due to the position of the weight, and

(2) the energy stored in the rod due to its deformation, i.e. strain energy.

In Fig. 18*a* the load-extension graph is given, the respective areas of the diagram representing the energy stored ; e.g. when the load is applied to the unstretched rod it causes an extension of amount δ_{st} and the energy stored

$$= \tfrac{1}{2}W \,.\, \delta_{st} = \triangle oab = \tfrac{1}{2}F \cdot \delta_{st}^2$$

since $W = F\delta_{st}$. This is the initial energy stored in the rod.

When the load is pulled down a distance x below its rest position, the total extension from the unstrained position is now $\delta_{st} + x$ and the force acting is $F(\delta_{st} + x) = ce$ on diagram.

$$\therefore \text{ Total energy stored in rod } = \tfrac{1}{2}F(\delta_{st} + x)^2 = \triangle oce$$
$$= \tfrac{1}{2}F\delta_{st}^2 + \tfrac{1}{2}Fx^2 + F \,.\, \delta_{st} \,.\, x.$$

But $F \,.\, \delta_{st} = W$;

$$\therefore \text{ Total energy stored} = \tfrac{1}{2}F\delta_{st}^2 + \tfrac{1}{2}Fx^2 + Wx.$$

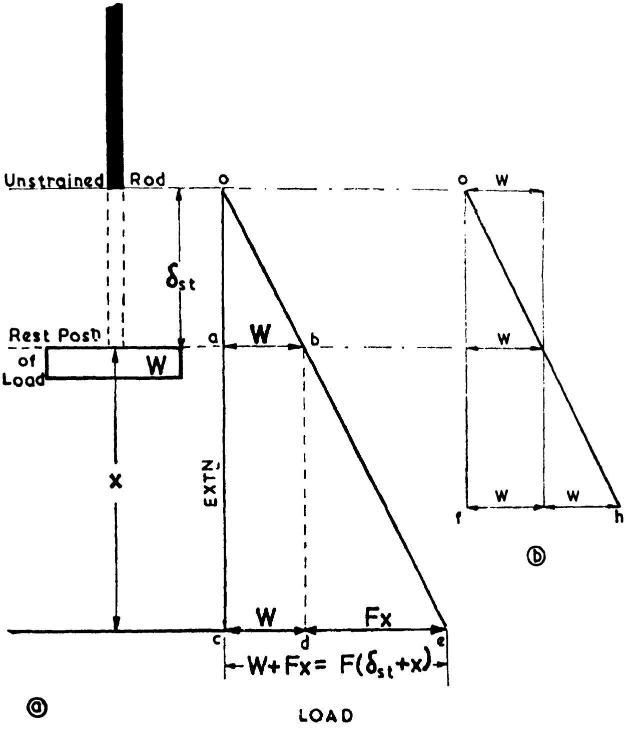

FIG. 18.

Now at the same time that this energy is being stored, the load W is losing potential energy by dropping down the distance x, i.e. the loss of potential energy $= Wx$.

Hence the total P.E. in this position is

$$\tfrac{1}{2}F\delta_{st}^2 + \tfrac{1}{2}Fx^2.$$

This we can see at once from the diagram:

$$\text{Area } acdb = Wx = \text{loss of P.E. of load.}$$

$$\therefore \text{ Total P.E.} = \triangle oce - acdb$$
$$= \triangle oab + \triangle bde$$
$$= \tfrac{1}{2}F\delta_{st}^2 + \tfrac{1}{2}Fx^2.$$

Thus the entire change in P.E. from the initial rest conditions is $\frac{1}{2}Fx^2$, *which is equivalent to that for an unloaded rod extended by the same amount* x, gravity therefore not affecting matters other than fixing the rest position to give the required static tension to balance the load.

The Kinetic Energy of the load W at any displacement x from its rest position $=\frac{1}{2}\frac{W}{g}\left(\frac{dx}{dt}\right)^2$.

Thus $\frac{1}{2}F\delta_{st}^2+\frac{1}{2}Fx^2+\frac{1}{2}\frac{W}{g}\left(\frac{dx}{dt}\right)^2=$constant.

But $\frac{1}{2}F\,.\,\delta_{st}^2$ is a constant, being the initial energy stored in the rod.

Hence we have

$$\frac{1}{2}\frac{W}{g}\left(\frac{dx}{dt}\right)^2+\frac{1}{2}Fx^2=\text{constant.} \quad\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Now while we have deduced this equation for a load on a rod, a similar equation will obviously hold for a load supported by any elastic structure which has one degree of freedom, i.e. as we have mentioned above, a system whose position can be given in terms of a single variable.

It will be seen that the energy equation then only contains this variable and its first differential coefficient.

If we differentiate both sides of the equation with regard to time, we obtain

$$\frac{W}{g}\cdot\frac{dx}{dt}\cdot\frac{d^2x}{dt^2}+Fx\,.\,\frac{dx}{dt}=0,$$

or

$$\frac{W}{g}\cdot\frac{d^2x}{dt^2}+Fx=0, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

which is the simple harmonic equation of motion deduced above.

(This is a simple particular case of Lagrange's method for determining the equation of motion of any system.)

Returning to our energy equation (1), in order to determine the value of the constant we must know the initial conditions, i.e. what displacement and velocity the body has when we start to measure time.

Suppose we pull the weight W down a distance x_0 and release it without giving it any impulse, i.e. the initial velocity is zero, then substituting in (1), we obtain

$$0+\frac{Fx_0^2}{2}=\text{const.}$$

Hence
$$\frac{W}{2g}\left(\frac{dx}{dt}\right)^2+\frac{Fx^2}{2}=\frac{Fx_0^2}{2}.$$

This means that the sum of the K.E. and P.E. of the system during its motion is equal to the strain energy of deformation from the rest position—a result we should expect from the conservation of energy, since this strain energy $\frac{1}{2}Fx_0^2$ is the only energy added at the start, the body being at rest in the displaced position x_0.

When the displacement $x=x_0$, then the K.E., and therefore the velocity, becomes zero, i.e. the body has reached the end of its path. Thus x_0 is the amplitude of the motion. This also is to be expected, since we start with a displacement x_0 and, as no energy is lost or gained, we should expect the body to return to the same point each time. The energy in this position is all P.E.

When $x=0$, the body is passing through its equilibrium position, and the velocity has its maximum value, the energy being now all K.E. Hence

$$\text{Maximum K.E.}=\tfrac{1}{2}Fx_0^2.$$

Thus, recapitulating, we see that if a body is given an initial displacement x_0 and released, the total energy possessed during vibration is equal to the energy of deformation of the elastic support, which appears wholly as P.E. at each end of a vibration, the amplitude being x_0, and wholly as K.E. as the body passes through its equilibrium position.

Live or dynamic loads. In the problem just considered confusion sometimes arises such as is expressed by the following question: Why, if the strain energy of the rod is measured from its unstretched position, is the change of P.E. of the load only measured from its rest position although the load was attached to the unstretched rod in the first place? (Fig. 18*a*).

To make this clear, we can imagine matters in two ways: (*a*) the load W when attached was not allowed to fall freely downwards, but was brought slowly to its rest position, this being the same as if a force, gradually increasing from zero to its maximum value W, were applied to the rod, for which state of affairs the $\triangle oab$ represents the energy stored.

Alternatively (b), if we imagine the load to drop freely when attached we have what is termed a " suddenly applied " or a " live load ", and it is clear that the loss of P.E. of the load in dropping to its rest position is $W \, . \, \delta_{st}$. But the energy stored in the rod is only half of this, i.e. $\triangle oab$, hence the load overshoots its rest position and will vibrate about this position indefinitely unless the excess energy is damped out by friction. Fig 18b shows these conditions. The load W is initially supported so that no force is applied to the rod. Upon sudden removal of the supports the load will fall until its loss of P.E., i.e. $W \times of$, is equal to the strain energy stored in the rod, $\triangle ofh$.

Thus
$$W \times of = \tfrac{1}{2} of \times fh \, ;$$
$$\therefore \; fh = 2W,$$
and
$$\therefore \; of = 2\delta_{st}.$$

The load will therefore vibrate about its rest position with amplitude $= \delta_{st}$, *and the maximum stress produced in the rod due to this sudden application of the load is twice the static stress.*

If the load had been dropped a distance h on to a stop at the end of the unstretched rod the distance of would have been such that
$$W(of + h) = \tfrac{1}{2} of \times fh,$$
and the maximum stress would have been correspondingly greater than $2W$.

Example : A vertical tie rod carries a load of 30,000 lb. by which it is stretched 0·72 in. If this load was to fall 0·3 in. on to a stop at the free end of the bar, with what period would the mass vibrate and what would be the amplitude?

The Period will be $2\pi\sqrt{\dfrac{\delta_{st}}{g}} = 2\pi\sqrt{\dfrac{0{\cdot}72}{32 \times 12}} = 0{\cdot}272$ sec.

For the amplitude we have

$$\text{Loss of P.E. of load} = 30{,}000(of + 0{\cdot}3) \text{ in. lb.}$$

The maximum force $P = fh$ can be obtained since 30,000 lb. produced an extension of 0·72 in.

∴ the extension of is produced by a force $P = 30{,}000 \times \frac{of}{0{\cdot}72}$;

∴ equating,

$$30{,}000\,(of + 0{\cdot}3) = \frac{1}{2}\cdot\frac{30{,}000\,.\,of}{0{\cdot}72}\times of;$$

$$\therefore\ of^2 = 1{\cdot}44\,of + 0{\cdot}432\,;$$

whence $of = 0{\cdot}72 \pm 0{\cdot}975$ in.

Thus 0·975 in. is the amplitude and the two values of of obtained are those for the top and bottom of the vibration measured from the end o of the unstretched rod. Note that at the top of the movement the weight will leave the stop if it is still loose after its initial fall, so that a true vibration will not occur.

To account for the inertia of the supports. So far with every system under consideration we have neglected the mass, or moment of inertia as the case may be, of the elastic constraints. In certain problems, however, this may not be negligible in comparison with that of the load and may considerably modify the motion from the simple harmonic that we have assumed.

In order to gain an idea as to the effect of the inertia of the supports, let us again take the case of the longitudinal vibrations of a single load at the free end of an elastic rod (Fig. 19). If we assume that the mass of the rod, while not negligible, is *small* compared with that of the load, then we shall not be far wrong in assuming that the displacements of every point in the system during vibration will be so little affected that we may consider them to be the same as for a massless rod.

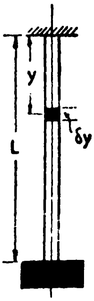

Fig. 19.

This means that the potential energy of both systems may be assumed to be approximately the same, since from the energy equation we see that the P.E. term depends only on the (displacement)^2 ; and hence the main effect will be on the K.E. term.

Let w = the weight per unit length of the rod.
y = the distance of an element δy of the rod when at rest from the fixed end, and
L = the length of the rod under the same conditions.

Then if x is the displacement of the load at the free end of the rod at any instant, the displacement of the element δy will be, on the above assumption, the same as for a rod of no mass, i.e. proportional to the distance from the fixed end, or

$$\text{Displacement of element} = x \cdot \frac{y}{L}.$$

Thus the velocity of the element will be

$$\frac{dx}{dt} \cdot \frac{y}{L} \quad \text{or} \quad V \cdot \frac{y}{L},$$

where $V = dx/dt$ is the velocity of the load at that instant.

Hence the K.E. of the element $= \frac{1}{2} \frac{w \cdot \delta y}{g} \left[V \cdot \frac{y}{L} \right]^2$;

∴ Total K.E. of whole system

$$= \frac{1}{2} \frac{W}{g} V^2 + \frac{1}{2} \frac{w}{g} V^2 \cdot \frac{1}{L^2} \int_0^L y^2 dy$$

$$= \frac{1}{2g} V^2 \left[W + \frac{wL}{3} \right],$$

which represents the K.E. of a load of weight $\left[W + \frac{wL}{3} \right]$ concentrated at the free end of the rod.

But wL = the weight of the rod, hence the inertia of the rod can be accounted for by adding $\frac{1}{3}$ of the mass of the rod to the mass of the load.

By exactly similar reasoning, but converting into angular motion, we should find that the inertia of the shaft in Case (1) for torsional oscillations may be included by adding $\frac{1}{3}$ of the polar moment of inertia of the shaft to the moment of inertia of the load.

In this manner, for systems of one degree of freedom, it is easy to allow for the inertia of the elastic support by considering the kinetic energy of the system, but in more complicated problems it is not so straightforward. This approximate method is due to Lord Rayleigh.

When the mass of the rod is comparable with that of the load, a more exact mathematical treatment is necessary, and the case of a marine propeller and its shafting is considered in a later chapter.

CHAPTER III

NATURAL VIBRATIONS OF SYSTEMS HAVING SEVERAL DEGREES OF FREEDOM

Longitudinal vibration of systems having several degrees of freedom. A system is said to have z degrees of freedom when z independent variables are required to specify its configuration at any instant.

We have so far considered the vibration of a system having one degree of freedom and have found that the motion is simple harmonic.

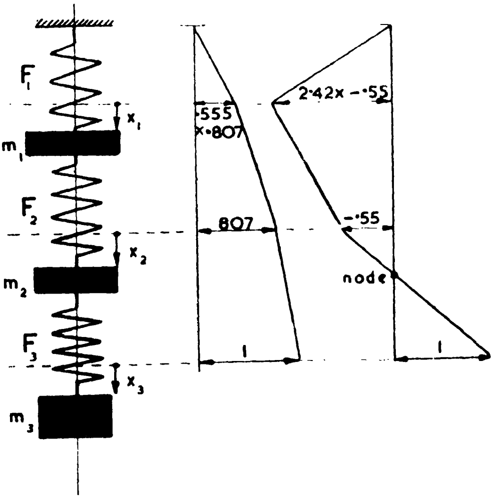

Fig. 20.

General equations of motion for a series of loads moving in any manner can be deduced but are outside the scope of this work ; however, in most practical cases we are concerned with a vibration of a particular type—longitudinal, angular, or transverse, etc., i.e. we have introduced constraints which reduce the number of degrees of freedom by confining the motion to a specified type, and in all such cases it can be shown that a particular solution of the differential equations of motion is simple harmonic in character.

In order to fix our ideas consider (say) the longitudinal vibration of the series of masses coupled together elastically as in Fig. 20. Here there will be three degrees of freedom since the configuration at any instant can be denoted by the displacements x_1, x_2, x_3 of the loads from their respective rest positions.

Then, as stated, it can be shown that one particular way in which the system may perform free vibrations is such that

$$x_1 = a_1 \cos (nt + e),$$

$$x_2 = a_2 \cos (nt + e),$$

$$x_3 = a_3 \cos (nt + e),$$

which means that *the loads move with simple harmonic motion, all reaching their maximum positions or passing through their means together.*

This is called a *natural or normal mode* of vibration, since it is the motion that the system most readily acquires, and this is the only type of motion that will concern us in the following pages.

We shall now show that for a given system there are a number of such normal modes of different frequencies.

Returning to the case of three loads (Fig. 20) :

Let x_1, x_2, x_3 = the displacements of the loads *from their rest positions* at any given instant, being measured positive downwards.

F_1, F_2, F_3 = the forces to produce unit alteration in length of each spring respectively, i.e. the stiffnesses.

Then considering load (1) moving down, this has two forces acting upon it upwards :

(*a*) The restoring force due to spring (1) $= -F_1 x_1$;

(*b*) The force due to the compression of spring (2) through $(x_1 - x_2) = -F_2(x_1 - x_2)$,

the alteration in length of a spring being the displacement of the end at the load concerned minus the movement of the other end.

Hence the equation of motion for this load is

$$\frac{W_1}{g}\frac{d^2x_1}{dt^2}=-F_1x_1-F_2(x_1-x_2). \quad\dots\dots\dots\dots\dots(1)$$

(If x_2 is greater than x_1, spring (2) is in tension and therefore exerts a downward force on m_1. This, of course, is automatically allowed for by sign in the above.)

Changing our datum to the rest position of load (2) and assuming this load to be moving downward, spring (2) will now be extended by (x_2-x_1) and exert an upward force $-F_2(x_2-x_1)$, and spring (3) a force $-F_3(x_2-x_3)$ in the same direction, being compressed by (x_2-x_3).

Thus $$\frac{W_2}{g}\frac{d^2x_2}{dt^2}=-F_2(x_2-x_1)-F_3(x_2-x_3). \quad\dots\dots\dots\dots(2)$$

Similarly, transferring to load (3) the extension of spring (3) is (x_3-x_2), and therefore

$$\frac{W_3}{g}\frac{d^2x_3}{dt^2}=-F_3(x_3-x_2). \quad\dots\dots\dots\dots\dots\dots\dots\dots(3)$$

Now making the assumption above that the system is vibrating in a normal mode, we have (using for convenience the "dot" notation $\ddot{x}$ for d^2x/dt^2):

$$\begin{aligned}
x_1&=a_1\cos(nt+e), &\therefore\ \ddot{x}_1&=-n^2a_1\cos(nt+e)=-n^2x_1;\\
x_2&=a_2\cos(nt+e), &\therefore\ \ddot{x}_2&=-n^2x_2;\\
x_3&=a_3\cos(nt+e), &\therefore\ \ddot{x}_3&=-n^2x_3;
\end{aligned}$$

where $\frac{n}{2\pi}$ is the frequency of the motion.

Thus substituting above, we obtain

$$(1)\ \frac{W_1}{g}n^2x_1=F_1x_1+F_2(x_1-x_2);$$

$$(2)\ \frac{W_2}{g}n^2x_2=F_2(x_2-x_1)+F_3(x_2-x_3);$$

$$(3)\ \frac{W_3}{g}n^2x_3=F_3(x_3-x_2).$$

[Note the interpretation of the equation having changed the signs: $\frac{W}{g}\ddot{x}$ or $-\frac{W}{g}n^2x$ is the force accelerating the load, $+\frac{W}{g}n^2x$ is the force exerted by the load.]

These three equations contain four unknowns x_1, x_2, x_3 and n^2, but only three independent variables since the displacements of any two of the loads can always be expressed as a ratio of the third, e.g. 1, x_2/x_1, x_3/x_1, so that the x's can be eliminated and we are left with an equation which for this case will be in $(n^2)^3$.

This is the *frequency equation.* There will thus be three roots n_1^2, n_2^2, n_3^2, all of which will be found to be real and unequal. This means that there are three ways or modes in which the system can vibrate with simple harmonic motion, each with a different frequency, given by $\frac{n_1}{2\pi}, \frac{n_2}{2\pi}, \frac{n_3}{2\pi}$.

The configuration of the system during any one of these normal modes of vibration can be determined from the three equations of motion of the loads by substituting the value of n^2 concerned and hence obtaining expressions for the ratios $\frac{x_1}{x_2}$ and $\frac{x_2}{x_3}$

If n_1^2, n_2^2, n_3^2 represent the roots in ascending order of magnitude it will be found that:

(*a*) Substituting n_1^2, these ratios are both positive. This means that the loads all move down or up together, i.e. their displacements are all on the same side of the mean at any instant.

(*b*) Substituting n_2^2, one ratio will be positive, the other negative.

This means that two adjacent loads will be moving down together (positive ratio) while the third load will be moving in the opposite direction (negative ratio).

Thus a node will be formed in the spring coupling the two loads whose displacements are in opposite directions.

(*c*) Substituting the highest value n_3^2, both ratios will be negative. Thus the centre load m_2 will be moving in the opposite direction to the outer loads m_1 and m_3. There will therefore be a node in both the springs (2) and (3).

In each of these natural modes of vibration the loads, of course, are moving in step with one another, all reaching their maximum displacements or passing through their mean positions together.

The slowest mode is usually termed the *fundamental* and possesses the least number of nodes, and each succeeding mode in ascending magnitude of frequency contains one more node. A node can only occur in an elastic connexion between two loads.

Let us work out a simple case of the above three-load system as an illustration.

We will assume that

$$\frac{W_1}{g} = \frac{W_2}{g} = \frac{W_3}{g} = \text{unity},$$

and
$$F_1 = F_2 = F_3 = \text{unity}.$$

(The above values are obviously absurd practically, but we are here concerned only with the interpretation of the solution.)

The equations of motion thus reduce to

$$(1)\quad (n^2 - 2)\, x_1 = -x_2\,;$$

$$(2)\quad (n^2 - 2)\, x_2 = -x_1 - x_3\,;$$

$$(3)\quad (n^2 - 1)\, x_3 = -x_2.$$

From (3)

$$x_3 = \frac{-x_2}{n^2 - 1},$$

substituting in (2)

$$x_2 \left(n^2 - 2 - \frac{1}{n^2 - 1} \right) = -x_1,$$

and then from (1) substituting for x_2

$$(n^2 - 2)\, .\, x_1 \left(n^2 - 2 - \frac{1}{n^2 - 1} \right) = x_1,$$

whence
$$(n^2)^3 - 5(n^2)^2 + 6n^2 - 1 = 0.$$

The roots may be found by a process of trial-and-error substitution and plotting, and are (very nearly)

$$n_1{}^2 = 0{\cdot}02,$$

$$n_2{}^2 = 1{\cdot}55,$$

$$n_3{}^2 = 3{\cdot}236.$$

Now from equations (1) and (3) we have

$$\frac{x_1}{x_2}=\frac{1}{(2-n^2)},\quad \frac{x_2}{x_3}=1-n^2,$$

and, substituting the above roots, these ratios become :

Root	x_1/x_2	x_2/x_3
$n_1{}^2$	$+0{\cdot}555$	$+0{\cdot}807$
$n_2{}^2$	$+2{\cdot}42$	$-0{\cdot}55$
$n_3{}^2$	$-1{\cdot}236$	$-2{\cdot}236$

Thus when the system is moving in the mode of frequency $n_2/2\pi$ a node is formed in spring (3) ; and when the vibration has a frequency $n_3/2\pi$ there is a node formed in both the springs (2) and (3). The fixed support, of course, always constitutes a node.

The normal elastic curves for the first and second modes of vibration are shown in Fig. 20, the amplitude of m_3 being taken as unity. The curves have been plotted along the static length of the system.

These solutions to the frequency equation are really " partial " solutions, the general complete solution being the sum of them all, i.e. the motion which will occur when any elastic system is displaced from its equilibrium position can be considered as produced by superimposing all or any of these normal modes of vibration. If, however, any one of these modes is started alone, the system will continue to vibrate with this simple harmonic motion. *The number of such normal modes is equal to the number of degrees of freedom* which the system possesses.

The general motion would therefore be

$$x_1=y_1\cos(n_1t+e_1)+y_2\cos(n_2t+e_2)+\dots,$$

$$x_2=z_1\cos(n_1t+e_1)+z_2\cos(n_2t+e_2)+\dots,$$

where y_1, z_1 ... , y_2, z_2 ... are the amplitudes of the masses in each of the normal modes respectively.

The resulting motion is of course *not* simple harmonic.

Consider any system of two masses, having two degrees of freedom, to which the above equations would apply, and suppose the arrangement is such that n_1 and n_2 are not very different in value. Then the motion of each mass is the sum of two simple harmonic motions of nearly the same frequency, so that in this case the resultant motion of each is approximately simple harmonic, and the amplitudes will alternately increase and diminish (beats). These amplitudes cannot both reach a maximum or a minimum together, because the system, being in free vibration, possesses a constant stock of energy (assuming no damping) so that as the energy of one mass is increased by increase in amplitude, that of the other must decrease.

Suppose, for example, that the displacement ratios for this two-mass system were

$$\frac{y_1}{z_1}=\frac{1}{2} \quad \text{and} \quad \frac{y_2}{z_2}=-1,$$

being obtained by substituting the two roots n_1^2, n_2^2 in the original equations of motion ; and suppose the system is released after having given mass m_1 a downward displacement of one inch, whilst m_2 was held in its rest position. This displacement given to m_1 will be, therefore, the maximum displacement it can attain, i.e. the vectors y_1, y_2 concerned start in their maximum positions, or, in other words, the phase angle $e_1=e_2=$ zero and so, with $t=0$,

$$x_1=1''=y_1+y_2,$$

while

$$x_2=0 \;=z_1+z_2,$$

whence, using the above ratios,

$$y_1=\tfrac{1}{3}'', \qquad y_2=\tfrac{2}{3}'' ;$$

$$z_1=\tfrac{2}{3}'', \qquad z_2=-\tfrac{2}{3}''.$$

Now the y_1 and z_1 motions are always in phase, being of frequency $n_1/2\pi$ (no node in connexion between masses, $y_1/z_1=+\text{ve}$), while the y_2 and z_2 motions are 180 degrees out of phase (node, $y_2/z_2=-\text{ve}$) and of frequency $n_2/2\pi$, which we are assuming to be not very different from the other. The y_2 motion will start in phase with the y_1 motion with maximum displacement, thus giving our initial condition that the mass m_1 starts with its maximum displacement $y_1+y_2=1$ inch. Since the two masses in both component motions start at their maximum displacements, their initial velocities are zero ; so that the values of y and z given satisfy the other conditions that both masses are initially at rest, m_1 in its maximum position and m_2 in its mean position. These results could have been deduced by substituting the initial conditions in the expressions for the displacements and velocities of the masses. These initial conditions for the two natural modes of a two-mass system similar to Fig. 20 are shown in Fig. 21(*a*) and (*b*) and their resultant in (*d*).

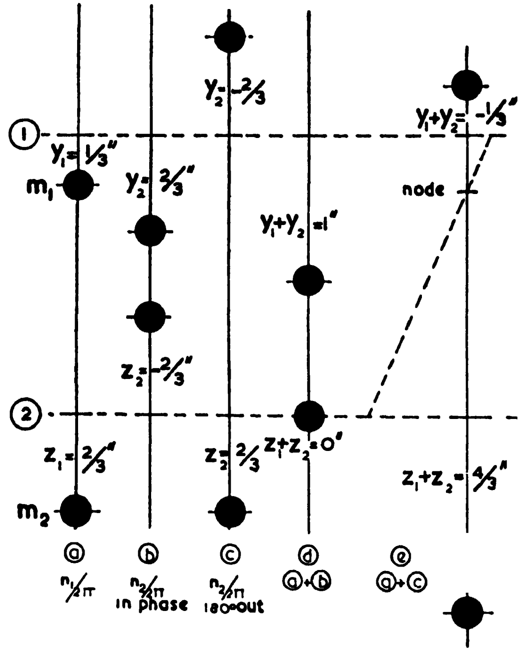

FIG. 21.

The amplitude of m_1 slowly reduces as the motions get out of phase, becoming $y_1+y_2=\frac{1}{3}+(-\frac{2}{3})=-\frac{1}{3}$ in the $\frac{1}{2}$ beat period. During this same time the amplitude of m_2, which started as zero, has become $z_1+z_2=\frac{2}{3}+(+\frac{2}{3})$, i.e. $+\frac{4}{3}$ (the positive and negative signs merely indicating the relative positions above or below the mean), and it will be seen that a node, which started originally at the mass m_2, has moved slowly from this position to a point which divides the connecting spring in the ratio of the displacements now occurring, i.e. $\frac{1}{3}/\frac{4}{3}$.

The process is now reversed.

Rather similar phenomena will be discussed in the next two sections, and again in the section dealing with combined linear and torsional oscillations.

Twin pendulums elastically coupled. Fig. 22 (*a*) shows two equal masses m_1, m_2 (of weight W) attached to light rigid links of the same length l, which are free to oscillate in the same

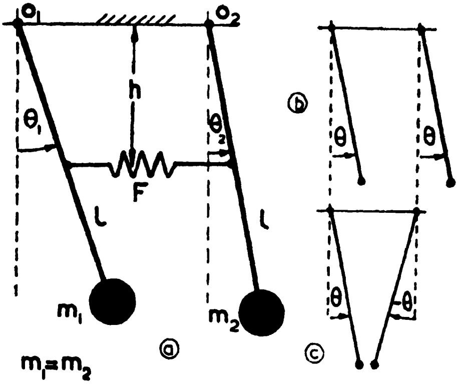

FIG. 22.

vertical plane about the points O_1, O_2. A spring of stiffness F connects the links at a distance h from the suspension points and is unstrained when the links are vertical.

Since the configuration is completely specified during vibration by the angular displacements θ_1 and θ_2, there are two degrees of freedom and therefore two natural modes. Assuming that the system can vibrate as shown we can write down the equations of motion :

Pendulum (1) : in addition to the restoring torque due to its own weight, the spring is compressed a distance $h(\theta_1-\theta_2)$ and also exerts a restoring torque $-F\,.\,h(\theta_1-\theta_2)\times h$

$$\therefore\ \frac{W}{g}l^2\frac{d^2\theta_1}{dt^2}=-Wl\theta_1-Fh^2(\theta_1-\theta_2).$$

Changing our datum to

Pendulum (2) : the spring is now stretched through $h(\theta_2-\theta_1)$ and exerts again a restoring torque $-Fh^2(\theta_2-\theta_1)$, so that

$$\frac{W}{g}l^2\frac{d^2\theta_2}{dt^2}=-Wl\theta_2-Fh^2(\theta_2-\theta_1).$$

Assuming that

$$\theta_1=a_1\cos(nt+e),$$
$$\theta_2=a_2\cos(nt+e),$$

the equations of motion reduce to

$$\frac{W}{g}\,l^2n^2\theta_1 = Wl\theta_1 + Fh^2(\theta_1 - \theta_2) \quad \ldots\ldots\ldots\ldots\ldots(1)$$

$$\frac{W}{g}\,l^2n^2\theta_2 = Wl\theta_2 + Fh^2(\theta_2 - \theta_1). \quad \ldots\ldots\ldots\ldots\ldots(2)$$

Dividing equation (1) by (2), we get

$$\frac{\theta_1}{\theta_2} = \frac{Wl\theta_1 + Fh^2(\theta_1 - \theta_2)}{Wl\theta_2 + Fh^2(\theta_2 - \theta_1)};$$

$$\therefore\ Wl\theta_1\theta_2 + \theta_2 Fh^2(\theta_1 - \theta_2) = Wl\theta_1\theta_2 + \theta_1 Fh^2(\theta_2 - \theta_1);$$

$$\therefore\ \theta_2\theta_1 - \theta_2^2 = \theta_1\theta_2 - \theta_1^2,$$

or

$$\frac{\theta_1^2}{\theta_2^2} = 1,$$

or

$$\frac{\theta_1}{\theta_2} = \pm 1.$$

Substituting $\theta_1 = \theta_2$ in (1) gives $n_1^2 = \frac{g}{l}$, which is the same as for a simple pendulum, both masses move in phase with the same amplitude (Fig 22*b*).

Substituting $\theta_1 = -\theta_2$ gives

$$n_2^2 = \frac{g}{l} + 2\,.\,\frac{Fg}{W}\,.\,\frac{h^2}{l^2},$$

the two masses moving in opposition and always symmetrically disposed (Fig. 22*c*).

Now if the coupling spring is very weak, i.e. F very small, the second term above is very small, and thus n_1 and n_2 differ by only a small amount. Hence in the general motion, when these two natural modes are superimposed, beats will be produced, and, following the same argument as in the preceding section, if pendulum (1) is given a displacement θ while pendulum (2) is held and the system is released from this configuration, we have

$$\theta_1 = y_1 + y_2 = \theta \quad \text{and} \quad y_1/z_1 = 1;$$

$$\theta_2 = z_1 + z_2 = 0 \qquad\qquad y_2/z_2 = -1;$$

whence

$$y_1 = y_2 = \frac{\theta}{2};$$

$$z_1 = -z_2 = \frac{\theta}{2},$$

and the amplitude of pendulum (1) will slowly change from $y_1+y_2=\theta$ to $y_1-y_2=0$, and in the same time the amplitude of pendulum (2) will increase from $z_1+z_2=0$ to $z_1-z_2=\theta$, so that each pendulum in turn comes to rest as the other reaches its maximum amplitude θ, energy being continually transferred from one to the other.

The double pendulum. This is illustrated in Fig. 23 (*a*) and consists of two heavy particles m_1, m_2 of weights W_1, W_2 suspended by light inextensible cords. Vibration is assumed to take place in one plane, hence there are two degrees of freedom.

To obtain the values of the frequencies of the possible modes of vibration we must use the general method of writing down the equations of motion of the masses. This problem is discussed in detail as the results deduced are interesting.

Let x and y be the horizontal displacements of the particles from the vertical rest position, being of the same sign when on same side of this rest position. These displacements, as for a simple pendulum, will be assumed to be small, hence the vertical movement of the particles will be negligible.

Also, for the two cords, let θ_1, θ_2 be their inclinations at a given instant, and T_1, T_2 their tensions.

Consider first the lower pendulum. The horizontal restoring force when m_2 is displaced a distance y is $T_2 \sin \theta_2 = T_2 \cdot \frac{y-x}{l_2}$, and, using the same reasoning as for the simple pendulum, we may assume that $T_2=W_2$ if the displacements are small, hence the equation of motion is

$$(2) \quad \frac{W_2}{g}\frac{d^2y}{dt^2} = -W_2 \cdot \frac{y-x}{l_2}.$$

Now as regards the first pendulum, since we are assuming the tension T_1 in the cord to be equal to the vertical force at its end, then approximately $T_1=(W_1+W_2)$.

Thus the horizontal forces acting on m_1 are :

(*a*) The horizontal component of this tension acting towards the mid position

$$=(W_1+W_2)\frac{x}{l_1}, \text{ and}$$

(*b*) The horizontal component of the tension in the lower cord acting outwards when the system has the configuration shown

$$=W_2\frac{(y-x)}{l_2}.$$

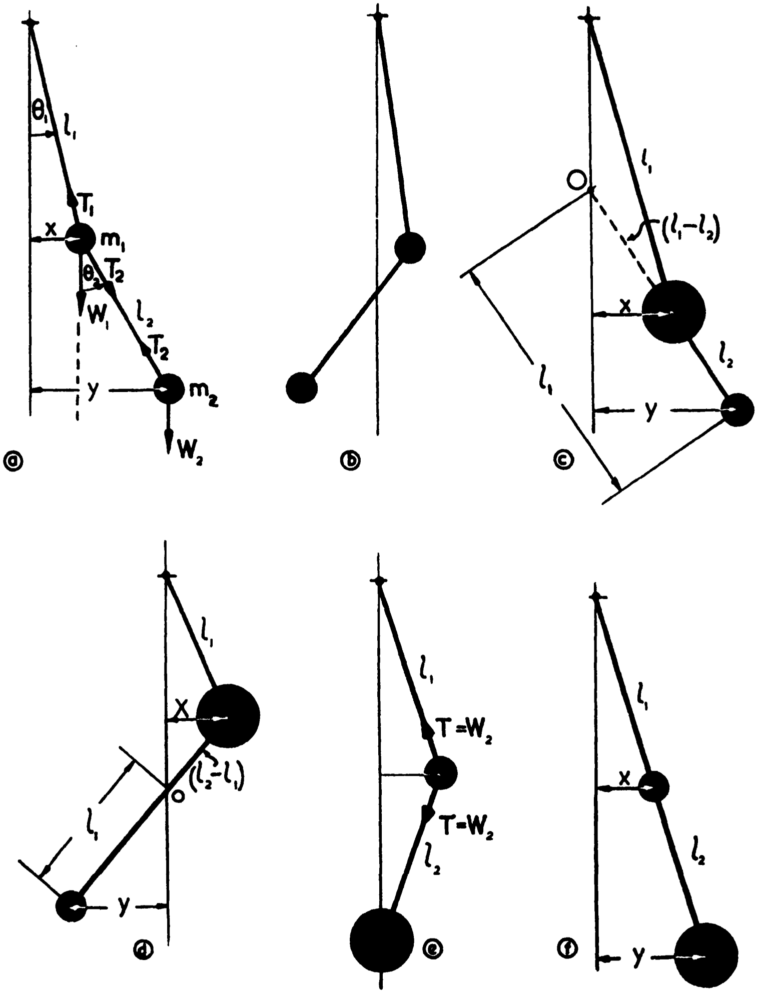

Fig. 23.

The equation of motion is therefore

$$(1)\quad \frac{W_1}{g}\frac{d^2x}{dt^2}=-(W_1+W_2)\frac{x}{l_1}+W_2\frac{(y-x)}{l_2}.$$

These equations will hold for all possible configurations upon substitution of the correct signs for x and y.

Assuming that simple harmonic vibrations can occur, let

$$x=a\cos(nt+e),\quad \therefore\ \frac{d^2x}{dt^2}=-n^2x;$$

$$y=b\cos(nt+e),\quad \therefore\ \frac{d^2y}{dt^2}=-n^2y.$$

Substituting above we obtain

$$(1)\ \left\{n^2-(1+R)\cdot\frac{g}{l_1}-R\cdot\frac{g}{l_2}\right\}x+R\cdot\frac{g}{l_2}\cdot y=0;$$

$$(2)\ \frac{g}{l_2}\cdot x+\left(n^2-\frac{g}{l_2}\right)y=0,\ \text{where } R=W_2/W_1.$$

From these equations we can eliminate x and y and leave an equation in $(n^2)^2$, e.g.:

From equation (1),

$$\frac{x}{y}=-R\frac{g}{l_2}\Big/\left\{n^2-(1+R)\frac{g}{l_1}-R\frac{g}{l_2}\right\},$$

and from (2),

$$\frac{x}{y}=-\frac{\left(n^2-\frac{g}{l_2}\right)}{\frac{g}{l_2}}.$$

Equating these expressions, we obtain

$$\left\{n^2-(1+R)\frac{g}{l_1}-R\cdot\frac{g}{l_2}\right\}\left(n^2-\frac{g}{l_2}\right)-R\left[\frac{g}{l_2}\right]^2=0,$$

or

$$(n^2)^2-(1+R)\left(\frac{g}{l_1}+\frac{g}{l_2}\right)n^2+(1+R)\frac{g}{l_1}\cdot\frac{g}{l_2}=0.\quad\text{...............(i)}$$

This is the "frequency equation", the two roots of which give the frequencies of the two normal modes of vibration.

In order to simplify matters let us assume

$$W_1=W_2,\ \text{i.e. } R=1,$$

and

$$l_1=l_2=l.$$

Equation (i) then becomes

$$(n^2)^2-4\frac{g}{l}\cdot n^2+2\left(\frac{g}{l}\right)^2=0$$

The roots of this quadratic are therefore given by

$$n^2 = \frac{\frac{4g}{l} \pm \sqrt{16\left(\frac{g}{l}\right)^2 - 8\left(\frac{g}{l}\right)^2}}{2}$$

$$= \frac{g}{l}\left[2 \pm \sqrt{2}\right].$$

Now from equation (2),

$$\frac{x}{y} = -\frac{\left(n^2 - \frac{g}{l}\right)}{g/l}, \qquad \text{(ii)}$$

and substituting the lower value of n^2, i.e.

$$n_1{}^2 = \frac{g}{l}\left[2 - \sqrt{2}\right],$$

we obtain

$$\frac{x}{y} = +0{\cdot}414.$$

Thus the masses are both on the same side of their mean position at any given instant.

For the higher frequency,

$$n_2{}^2 = \frac{g}{l}\left[2 + \sqrt{2}\right],$$

and hence $\frac{x}{y} = -2{\cdot}414.$

Thus the masses are on opposite sides of their mean positions, as in Fig. 23*b*.

There are some interesting special cases of the system worthy of note.

Suppose that $W_2/W_1 = R$ is very small, then the frequency equation (i) becomes *approximately*

$$(n^2)^2 - \left(\frac{g}{l_1} + \frac{g}{l_2}\right)n^2 + \frac{g}{l_1}\cdot\frac{g}{l_2} = 0, \qquad \text{(iii)}$$

the roots of which are $\frac{g}{l_1}$ and $\frac{g}{l_2}$ and the corresponding frequencies

$$\frac{1}{2\pi}\sqrt{\frac{g}{l_1}}, \quad \frac{1}{2\pi}\sqrt{\frac{g}{l_2}}$$

which are the same as for the two pendulums if separate. Thus in the mode corresponding to the first frequency, the upper pendulum vibrates just as if it were alone. This we could picture to be the case since m_1 is very heavy, and m_2, the particle suspended from it, is very light and will not appreciably affect the motion.

As regards the lower pendulum in this mode we can consider its support (i.e. m_1) to be given a simple harmonic motion of frequency $\frac{1}{2\pi}\sqrt{\frac{g}{l_1}}$ which will *force* the pendulum to take up the same frequency, while its natural frequency, i.e. with support stationary, would be

$$\frac{1}{2\pi}\sqrt{\frac{g}{l_2}}.$$

We are here, however, slightly in advance of our subject. This system described is dealt with in detail on page 168 and the reader is asked to refer back to this problem later.

Upon substituting $n^2 = \frac{g}{l_1}$ in equation (ii), we obtain

$$\frac{x}{y} = -\frac{l_2 - l_1}{l_1} = \frac{l_1 - l_2}{l_1}.$$

If $l_1 > l_2$ this means that the lower pendulum is moving in exactly the same manner as a simple pendulum of length l_1 suspended from the point O, which can be considered as an imaginary node, Fig. 23*c*.

If $l_1 < l_2$ then the ratio is negative and we have the result shown in Fig. 23*d*, the lower pendulum moving as if supported at O, which forms a real node.

In the second normal mode of frequency, $\frac{1}{2\pi}\sqrt{\frac{g}{l_2}}$ (approx.), the ratio x/y is zero, i.e. $x = 0$, so that the upper particle is at rest and the lower one swings as a simple pendulum with this frequency.

Now on page 58 we stated that the roots of a frequency equation are really partial solutions, and although any system, if once set into vibration in a particular mode, will continue to vibrate in that manner, yet the general motion produced when the system is disturbed will be the sum of all the normal modes existing.

Here, then, the general motion of the two particles will be obtained by superimposing the two modes given by the roots n_1, n_2, i.e.

$$x = a_1 \cos (n_1 t + e_1) + a_2 \cos (n_2 t + e_2),$$

and
$$y = b_1 \cos (n_1 t + e_1) + b_2 \cos (n_2 t + e_2).$$

If, as above, R is small and we make $l_1 = l_2$, the roots of equation (iii) become equal, so that the roots of the true equation (i) are approximately equal; and we have seen that the motion of each particle, being the resultant of two S.H.M.s of nearly equal frequencies, is itself approximately simple harmonic with an alternately increasing and diminishing amplitude.

If the amplitudes of the component motions are made equal, i.e. $a_1 = a_2 = a$ and therefore $b_1 = b_2 = b$, the effect is most marked —the resultant amplitudes of m_1 and m_2 varying from zero to $2a$ and $2b$ respectively, i.e. there are periods of approximate rest.

Since the system is in free vibration and no external forces act upon it, the energy which it possesses will be constant, so that m_1 and m_2 cannot be at rest together, but as the motion of one dies down the other must increase in amplitude, energy being alternately transferred from one particle to the other.

Now suppose the lower particle is much heavier than the upper, i.e. R is large. Let us see first whether we can visualise the nature of the motion.

The lower particle, being heavy, will tend to keep the whole cord taut in a straight line of length $l_1 + l_2$, the motion being approximately that of a simple pendulum of this length, the upper particle for all practical purposes not affecting matters.

A second way in which the system might vibrate can be deduced as follows. The lower particle will possess considerable inertia, hence when a given impulse is applied to start the motion the upper particle will probably respond more readily than the lower, which will remain approximately stationary, and the motion will be equivalent to that of a particle m_1 on a cord of length $l_1 + l_2$ which is stretched between two fixed points.

With R large, the frequency equation (i) becomes approximately :

$$(n^2)^2 - R\left(\frac{g}{l_1}+\frac{g}{l_2}\right)n^2 + R\,.\,\frac{g}{l_1}\,.\,\frac{g}{l_2}=0.$$

[Digressing for a moment :

Consider the equation

$$ax^2+bx+c=0.$$

If α, β are its roots we may write

$$a(x-\alpha)(x-\beta)=0,$$

i.e. $$ax^2-(\alpha+\beta)x+\alpha\beta=0.$$

Comparing this with the original, we see that

$$\text{Sum of roots } (\alpha+\beta) = -\frac{b}{a}.$$

$$\text{Product of roots } \alpha\beta = +\frac{c}{a}.$$

$$\text{Sum of reciprocals } \frac{1}{\alpha}+\frac{1}{\beta}=\frac{\alpha+\beta}{\alpha\beta}=-\frac{b}{c},$$

while the roots themselves are given by

$$\left.\frac{-b\pm\sqrt{b^2-4ac}}{2a}\right].$$

The roots of the above approximate frequency equation are, therefore,

$$\frac{1}{2}\left[R\left(\frac{g}{l_1}+\frac{g}{l_2}\right)\pm\sqrt{R^2\left(\frac{g}{l_1}+\frac{g}{l_2}\right)^2-\frac{4Rg^2}{l_1l_2}}\right]. \quad \text{.........(iv)}$$

The second term under the root sign is, however, very small compared with the first, e.g. the expression expanded is

$$\sqrt{R^2\left(\frac{g}{l_1}\right)^2+R^2\left(\frac{g}{l_2}\right)^2+\frac{2Rg^2}{l_1l_2}(R-2)},$$

and as R is large, the final term is approximately $\frac{2R^2g^2}{l_1l_2}$, so that the expression reduces to

$$R(g/l_1+g/l_2) \text{ very nearly.}$$

Thus, using the positive sign in (iv), one root is

$$R(g/l_1+g/l_2) \text{ approximately.} \quad \text{..................(v)}$$

If we use the negative sign, the other root would apparently be zero, but as this cannot be, its value can be determined from

$$\text{Products of roots} = \frac{R \, . \, g^2}{l_1 l_2},$$

whence, dividing this by (v), the second root is $\frac{g}{l_1 + l_2}$. ......(vi)

Using the first root we see that

$$\frac{x}{y} = -\frac{R(g/l_1 + g/l_2) - g/l_2}{g/l_2}$$

$$= -\frac{Rg/l_1 + g/l_2(R-1)}{g/l_2},$$

and, R being large, this becomes approximately

$$-R\frac{(l_1 + l_2)}{l_1}.$$

Hence x is seen to be large compared with y, and as, by hypothesis, the displacement of m_1 is small, the particle m_2 is almost stationary.

If we use the above results, that W_1 is negligible in comparison with W_2 and y with x, in conjunction with the original equation of motion of m_1, i.e. equation (1), this becomes

$$\frac{W_1}{g}\frac{d^2x}{dt^2} = -W_2\left(\frac{1}{l_1} + \frac{1}{l_2}\right)x,$$

which is the equation for the simple harmonic motion of a *light* particle m_1 attached to a *tightly* stretched cord of length $(l_1 + l_2)$ in which the tension is uniform and equal to W_2, e.g. Fig. 23*e*: if the displacement x is small so that the increase in tension produced in the cord is negligible in comparison with the original tension $T = W_2$, the horizontal restoring force is

$$-W_2\left(\frac{x}{l_1} + \frac{x}{l_2}\right).$$

The frequency is therefore $\frac{1}{2\pi}\sqrt{g \, . \, R\left(\frac{1}{l_1} + \frac{1}{l_2}\right)}$ which agrees with the root obtained above (v).

Substituting the second root (vi), i.e. $\frac{g}{l_1+l_2}$, in the expression for x/y obtained from equation (2) page 65, we get

$$\frac{x}{y}=\frac{l_1}{l_1+l_2},$$

which shows that the two lengths of cord keep in the same straight line during this mode, the system being approximately that of a simple pendulum m_2 on a length of cord l_1+l_2, the light particle m_1 not appreciably affecting matters, Fig. 23*f*.

Torsional oscillation of shaft systems carrying concentrated flywheel loads. The determination of the natural frequencies of torsional oscillation of shaft systems is one of the most important problems met with in practice, so that in this section the various methods are discussed in detail. The same methods can be employed for the determination of the natural frequencies of similar systems in longitudinal oscillation and some have been dealt with already.

(A) **The method of nodes.** In the previous chapter we saw that with two loads at the ends of a shaft free to revolve a node was formed which divided the system into two sections having the same frequency. Consider, now, a similar system carrying three loads :

Case (6). *Revolving shaft system carrying three loads* (Fig. 24*a*).

Here it will be seen that two modes of vibration are possible, there being two degrees of freedom, since the configuration during vibration is completely specified by the angular displacements of the loads expressed in terms of any one displacement :

(*a*) *With one node,* the centre and one outside load (which pair depends on the stiffness of the connexions and the magnitude of the moments of inertia) moving together in the opposite direction to the remaining load.

(*b*) *With two nodes,* the centre load moving against the two outer ones.

A node, other than a fixing, can be formed in any shaft connecting two loads. Let us imagine that all the nodes possible are formed, so that, here, a node will be formed in shaft section (1) at a distance x from I_1, and a second node in section (2) at a distance y from I_2.

The system is thus split up into simpler systems as in Fig. 24*b*, which we have already discussed in Chapter II, and, the frequency of the whole system being equal to that of any part, we have

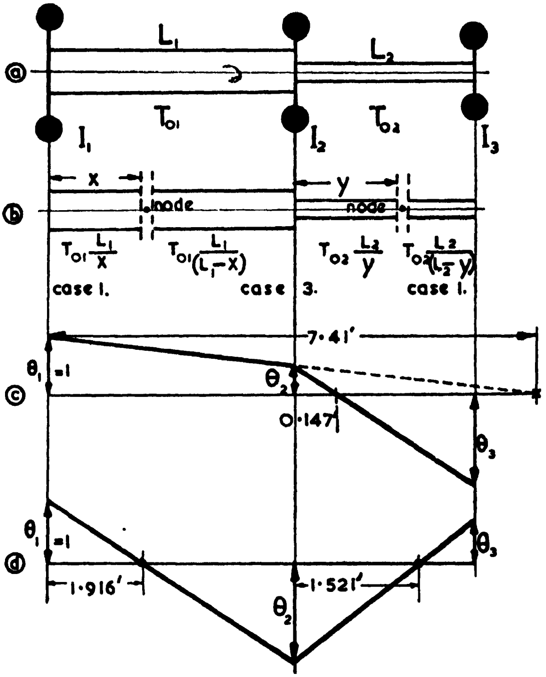

Fig. 24.

$$\text{frequency}=\underset{\text{(i)}}{\frac{1}{2\pi}\sqrt{\frac{g}{I_1}\left(T_{01}\frac{L_1}{x}\right)}}=\underset{\text{(ii)}}{\frac{1}{2\pi}\sqrt{\frac{g}{I_2}\left[T_{01}\cdot\frac{L_1}{L_1-x}+T_{02}\cdot\frac{L_2}{y}\right]}}$$

$$=\underset{\text{(iii)}}{\frac{1}{2\pi}\sqrt{\frac{g}{I_3}\left(T_{02}\frac{L_2}{L_2-y}\right)}},$$

from which a quadratic equation in x or y is obtained, giving two real values of x with a corresponding real and imaginary value of y (or vice versa), which represent the two possible modes of vibration.

An example will illustrate the above procedure :

A gas engine has a crankshaft 3 in. diam. and 4 ft. long, and carries two flywheels, both of moment of inertia 1 ton ft.^{2}, at its ends. An extension of the shaft of diam. 1·5 in. and length 2 ft. carries the armature of a dynamo of moment of inertia $\frac{1}{4}$ ton ft.^{2}

Neglecting the effects of the crank, determine the frequency of the two possible modes of natural torsional oscillation. Take the modulus of rigidity $C=5{,}000$ tons per sq. in.

Referring to Fig. 24, we here have

$$I_1=I_2=1 \text{ ton ft.}^2. \quad J_1=\frac{\pi(\frac{1}{4})^4}{32} \text{ ft.}^4$$

$$I_3=\tfrac{1}{4} \text{ ton ft.}^2$$

$$L_1=4 \text{ ft.} \qquad J_2=\frac{\pi(\frac{1}{8})^4}{32} \text{ ft.}^4$$

$$L_2=2 \text{ ft.}$$

$$C_1=C_2=5{,}000 \text{ tons per sq. in.}$$

Using ton, ft. units, we have

$$T_{01}=\frac{C_1 J_1}{L_1}=\frac{5{,}000\times 144\times\pi(\frac{1}{4})^4}{4\times 32} \text{ ton ft. per radian ;}$$

$$T_{02}=\frac{C_2 J_2}{L_2}=\frac{T_{01}}{8}.$$

Equating frequency expressions (i) and (iii),

$$\frac{T_{01}}{I_1}\cdot\frac{L_1}{x}=\frac{T_{02}}{I_3}\cdot\frac{L_2}{L_2-y};$$

$$\therefore\ \frac{T_{01}}{1}\cdot\frac{4}{x}=\frac{T_{01}/8}{\frac{1}{4}}\cdot\frac{2}{2-y};$$

$$\therefore\ \underline{y=2-x/4.}$$

Substituting, say, for y in (ii) and equating to (i),

$$\frac{T_{01}}{I_1}\cdot\frac{L_1}{x}=\frac{1}{I_2}\left[T_{01}\frac{L_1}{L_1-x}+T_{02}\cdot\frac{L_2}{y}\right],$$

i.e.

$$T_{01}\cdot\frac{4}{x}=\left[T_{01}\frac{4}{4-x}+\frac{T_{01}}{8}\cdot\frac{2}{2-x/4}\right].$$

Whence

$$9x^2-84x+128=0\ ;$$

$$\therefore\ x=7{\cdot}41 \quad\text{or}\quad 1{\cdot}916 \text{ ft.},$$

and

$$y=0{\cdot}147 \quad\text{or}\quad 1{\cdot}521 \text{ ft.}$$

When $x=7{\cdot}41$ ft. there is no real node in that shaft (L_1 being 4 ft.), the node occurring in the second section at $y=0{\cdot}147$ ft.

The meaning of the imaginary value is that the frequency of the whole system is equal to that of I_1 on a shaft of diameter 3 in. and length 7·41 ft. $\left(\text{stiffness } T_{01}\cdot\frac{L_1}{x}\right)$, so that if the N.E.C. between I_1 and I_2 were continued it would cut the axis at a distance 7·41 ft. from I_1.

This is the one node mode of vibration.

Substituting for x in Case (i)

$$\text{frequency}=\frac{1}{2\pi}\sqrt{\frac{5{,}000\times 144\times\pi(\frac{1}{4})^4\times 4\times 32}{4\times 32\times 7{\cdot}41\times 1}}\ \text{(ton, ft. units)}$$

$$=\underline{5{\cdot}50\ \text{cycles per sec.}}$$

When $x=1{\cdot}916$ and $y=1{\cdot}521$ a real node appears in both shafts, i.e. the two node mode of vibration, and substituting this value of x in Case (i) above we get

$$\text{frequency}=\underline{10{\cdot}80 \sim \text{per sec.}}$$

Knowing the position of the nodes, real or imaginary, and taking unit amplitude say for I_1, the relative amplitudes of I_2 and I_3 are at once obtained (Figs. 24*c* and *d*).

Thus, in general, to solve for the frequencies by the method of nodes—assume all nodes possible at distances x, y, z, etc., from the respective loads ; this divides the system into simple systems each having the same frequency. Sufficient equations can be found to eliminate the y's and z's, etc., and leave an equation in x^m where m is the number of modes possible.

Upon substituting the values of x so found in the respective frequency expression all the frequencies are obtained. The number and position of the nodes are determined from the values of x, y, etc.—if less than the corresponding length of the shaft section a real node exists.

Note that in some problems, when solving (say) for x, a *negative* value may be obtained, e.g. for the system shown in Fig. 25 the normal elastic curves for the two modes of vibration are shown in (*a*) and (*b*), there being no node and one node respectively in the shaft connecting the two loads. Having chosen to measure x from I_1 as shown, it so happens that in (*a*) the N.E.C. between I_1 and I_2 when produced cuts the axis to the left of I_1, giving a negative value therefore for x. To obtain the frequency substitute this negative value in the frequency expression involving x as before.

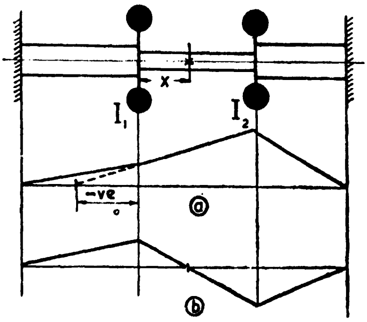

Fig. 25.

Whilst this method — of equating the frequencies of the simple cases formed by introducing a node in each shaft connecting two loads — is simple, yet it will be realised that with a large number of loads the mathematical work involved in the elimination will become very laborious, and even when the nodal equation has been obtained, this must be solved by a tedious process of trial and error.

In practice, however, many systems can be reduced to the case of three loads, e.g. a marine Diesel engine shafting, comprising propeller, flywheel and engine crank masses. If the connecting shafts are long it will be sufficiently accurate to assume that all the engine masses are concentrated in the centre plane of the engine.

(B) Determination of the natural frequencies of torsional oscillation of shaft systems, carrying concentrated loads, from the equations of motion of the loads. We have already studied this method in connexion with masses in longitudinal oscillation, so that we need only illustrate the method by studying some of the systems previously examined.

(*a*) *Shafting fixed at one end and carrying a series of loads.* Two loads are shown in Fig. 26.

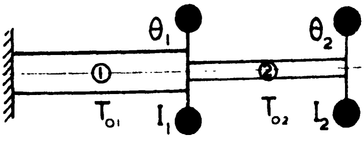

Fig. 26.

The following notation and symbols will be used throughout, each section of shafting between two loads or up to a change in diameter being numbered consecutively.

Let

I_1, I_2 = the moments of inertia of the loads in absolute units.

θ_1, θ_2 = the angular displacements of the loads at any given instant.

T_{01}, T_{02} = the torques to produce unit angle of twist in each shaft section respectively.

We will also use the abbreviated form $\ddot{\theta}$ for $\frac{d^2\theta}{dt^2}$.

Consider load (1) on its retaining shafts (1) and (2) : Shaft (1) is twisted through angle θ_1, and shaft (2) through angle $(\theta_1 - \theta_2)$, hence the restoring torque exerted on the load by the shafts

$$= -[T_{01}\theta_1 + T_{02}(\theta_1 - \theta_2)],$$

or the equation of motion is

$$\frac{I_1}{g}\ddot{\theta}_1 = -[T_{01}\theta_1 + T_{02}(\theta_1 - \theta_2)]. \quad \text{..............(1)}$$

Transferring now to load (2), we see that its retaining shaft, i.e. shaft (2), is twisted through an angle $(\theta_2 - \theta_1)$—note the change in sign of the angle since we have changed our datum—thus

$$\frac{I_2}{g}\ddot{\theta}_2 = -T_{02}(\theta_2 - \theta_1). \quad \text{....................(2)}$$

Assuming, as before, that every load vibrates in phase with S.H.M. of frequency $\frac{n}{2\pi}$, we may write:

$$\theta_1 = a_1 \cos(nt + e), \quad \therefore \ddot{\theta}_1 = -a_1 n^2 \cos(nt + e) = -n^2\theta_1;$$

$$\theta_2 = a_2 \cos(nt + e), \quad \therefore \ddot{\theta}_2 = -a_2 n^2 \cos(nt + e) = -n^2\theta_2.$$

Hence, substituting above,

$$\frac{I_1}{g}n^2\theta_1 = T_{01}\theta_1 + T_{02}(\theta_1 - \theta_2), \quad \text{.................(3)}$$

$$\frac{I_2}{g}n^2\theta_2 = T_{02}(\theta_2 - \theta_1). \quad \text{.........................(4)}$$

Thus by elimination of θ_1 and θ_2 (which can be done since there are only two independent variables n^2 and θ_1/θ_2, one displacement always being able to be expressed as a ratio of the other), we obtain our equation in $(n^2)^2$, the two roots of which give the two possible frequencies $\frac{n_1}{2\pi}, \frac{n_2}{2\pi}$.

From equation (4):

$$\theta_2 = \frac{T_{02}\theta_1}{T_{02} - \frac{I_2}{g}n^2}.$$

Substituting in (3) we obtain

$$\frac{I_1}{g}n^2 = T_{01} + T_{02} - \frac{T_{02}^2}{T_{02} - \frac{I_2}{g}n^2};$$

$$\therefore \frac{I_1}{g}\frac{I_2}{g}(n^2)^2 - n^2\left[\frac{I_1}{g}T_{01} + \frac{I_2}{g}(T_{01} + T_{02})\right] + T_{01}T_{02} = 0,$$

from which the two roots $(n_1)^2$, $(n_2)^2$ are obtained.

By substituting the roots in equations (3) or (4) we shall obtain the ratios of the amplitudes θ_1/θ_2 for each mode. One ratio will be found to be positive, i.e. there is no node in shaft (2), while the other is negative which means that the loads are moving in opposite directions, thus a node will be produced in shaft (2).

(*b*) *Case* (4) *above—two loads at the end of a shaft free to revolve* (Fig. 16).

Here then, the equations of motion will be

$$\frac{I_1}{g}\ddot{\theta}_1 = -T_{01}(\theta_1 - \theta_2),$$

$$\frac{I_2}{g}\ddot{\theta}_2 = -T_{01}(\theta_2 - \theta_1).$$

Substituting $\ddot{\theta}_1 = -n^2\theta_1$ for S.H.M.

$$\ddot{\theta}_2 = -n^2\theta_2,$$

we obtain

$$\frac{I_1}{g}n^2\theta_1 = T_{01}(\theta_1 - \theta_2), \quad \text{.......................(1)}$$

$$\frac{I_2}{g}n^2\theta_2 = T_{01}(\theta_2 - \theta_1). \quad \text{.......................(2)}$$

Eliminating θ_1 and θ_2, from (2)

$$\theta_1 = \left(1 - \frac{I_2 n^2}{g T_{01}}\right)\theta_2.$$

Substituting in (1)

$$\frac{I_1}{g} n^2 \left(1 - \frac{I_2 n^2}{g T_{01}}\right)\theta_2 = T_{01}\theta_2\left[\left(1 - \frac{I_2 n^2}{g T_{01}}\right) - 1\right];$$

$$\therefore\ \frac{I_1}{g} n^2 \left(1 - \frac{I_2 n^2}{g T_{01}}\right) = -\frac{I_2}{g} n^2;$$

$$\therefore\ I_1\left(1 - \frac{I_2 n^2}{g T_{01}}\right) = -I_2;$$

$$\therefore\ n^2 = \left(\frac{1}{I_2} + \frac{1}{I_1}\right) g \,.\, T_{01}$$

as previously deduced. The position of the node will be found as usual by substituting this value in equations (1) or (2), thus obtaining the ratio θ_1/θ_2, which will be negative.

(*c*) *Two loads at the ends of a shaft free to revolve, the shaft having sections of different diameters.*

Two such sections are shown in Fig. 27. Although it would be much simpler to obtain the equivalent stiffness of the shafting, or convert to an equivalent length of uniform shaft, we will here work out the problem as it stands since the method of procedure is of importance.

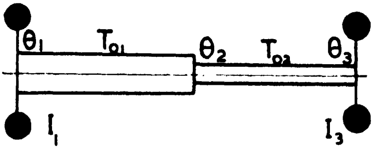

FIG. 27.

Let the loads be I_1, I_3, their angular displacements θ_1, θ_3, and let the angular displacement at the junction of the two shafts be θ_2 as indicated.

Then we have Load (1) :

$$\frac{I_1}{g}\ddot{\theta}_1 = -T_{01}(\theta_1 - \theta_2).$$

Now at the junction of the two shafts the torque exerted by section (2) on section (1) is equal and opposite to that by section (1) on section (2), i.e.

$$T_{01}(\theta_2 - \theta_1) = -T_{02}(\theta_2 - \theta_3).$$

Then finally

$$\frac{I_3}{g}\ddot{\theta}_3 = -T_{02}(\theta_3 - \theta_2).$$

Substituting $\ddot{\theta}_1 = -n^2\theta_1$ for S.H.M.

$$\ddot{\theta}_3 = -n^2\theta_3,$$

we have

$$\frac{I_1}{g} n^2\theta_1 = T_{01}(\theta_1 - \theta_2), \quad \ldots\ldots(1)$$

$$T = T_{01}(\theta_2 - \theta_1) = T_{02}(\theta_3 - \theta_2), \quad \ldots\ldots(2)$$

$$\frac{I_3}{g} n^2\theta_3 = T_{02}(\theta_3 - \theta_2). \quad \ldots\ldots(3)$$

Eliminating θ_1, θ_2, θ_3 we have:

From (2)

$$(T_{01} + T_{02})\,\theta_2 = T_{02}\theta_3 + T_{01}\theta_1.$$

Also

$$\frac{I_1}{g} n^2\theta_1 = -\frac{I_3}{g} n^2\theta_3,$$

i.e.

$$\frac{\theta_1}{\theta_3} = -\frac{I_3}{I_1}.$$

Thus, introducing these results in equation (1), we have

$$\frac{I_1}{g} n^2\theta_1 = T_{01}\theta_1 - T_{01}\left[\frac{T_{02}\theta_3 + T_{01}\theta_1}{T_{01} + T_{02}}\right]$$

$$= T_{01}\theta_1 - T_{01}\theta_1\left[\frac{-T_{02}\cdot\frac{I_1}{I_3} + T_{01}}{T_{01} + T_{02}}\right];$$

$$\therefore\ \frac{I_1}{g} n^2 = T_{01}\left\{1 - \frac{-T_{02}\cdot\frac{I_1}{I_3} + T_{01}}{T_{01} + T_{02}}\right\},$$

whence

$$n^2 = g\,\frac{T_{01}\cdot T_{02}}{T_{01} + T_{02}}\left(\frac{1}{I_1} + \frac{1}{I_3}\right)$$

$$= g\,\frac{\left(\frac{1}{I_1} + \frac{1}{I_3}\right)}{\frac{1}{T_{01}} + \frac{1}{T_{02}}}.$$

From a study of these simple cases the student should experience no difficulty in writing down the equations for any system, but it will be realised that the process of elimination of the θ's becomes very laborious when the number of loads is large.

The most important case of all is that of a shaft system free to revolve, and if the student were to work out the frequency equations with (say) three and four loads, he would notice that the terms involved follow an ordered arrangement such that it should be possible to evolve a general expression for any number of loads. This has been done by various investigators, and such formulae will be found in more advanced works and save the preliminary work of elimination. The solution of the equation, being a matter of trial and error, will still be tedious.

We will now consider a second method for obtaining the frequency equation which, with a tabulated system of trial and error solution, is eminently suited to practical problems. A full discussion of this method is given by Professor F. M. Lewis in his paper " Torsional Vibration in the Diesel Engine ".*

(C) Tabulated system of trial and error solution for the frequencies of torsional oscillation of a shaft system which carries concentrated loads and is free to revolve. We have seen that the torque which must be applied to a load of moment of inertia I absolute units to give it a S.H.M. of frequency $\frac{n}{2\pi}$ is

$$\frac{I}{g}\ddot{\theta} = -\frac{I}{g}n^2\theta.$$

This is the torque exerted on the load by its elastic constraint, so that the torque exerted by the load is equal and opposite, i.e.

$$+\frac{I}{g}n^2\theta.$$

Consider the first load of the system shown in Fig. 28, and, for convenience, assume $\theta_1 = \text{unity}$.

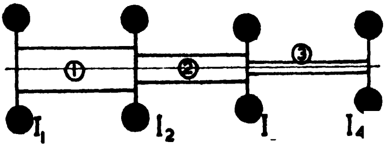

FIG. 28.

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* *Trans. Soc. American Naval Architects and Marine Engineer* 925.

Then :

Torque exerted by load (1) on shaft (1) $=\frac{I_1}{g}n^2 \times 1$,

$\therefore$ Twist produced in shaft (1) $=(\theta_1-\theta_2)=\frac{I_1}{g}n^2 \times \frac{1}{T_{01}}$;

$\therefore$ Amplitude of load (2) $\theta_2=\theta_1-(\theta_1-\theta_2)=\left[1-\frac{I_1}{g}\cdot\frac{n^2}{T_{01}}\right]$;

$\therefore$ Torque exerted by load (2) $=\frac{I_2}{g}n^2\left[1-\frac{I_1}{g}\frac{n^2}{T_{01}}\right]$;

$\therefore$ Torque transmitted to shaft (2) $=$ sum of torques due to loads (1) and (2)

$$=\left\{\frac{I_1}{g}n^2+\frac{I_2}{g}n^2\left[\quad\right]\right\};$$

$\therefore$ Twist in shaft (2) $=(\theta_2-\theta_3)=\frac{1}{T_{02}}\left\{\quad\right\}$;

$\therefore$ Amplitude of load (3) $=\theta_3=\theta_2-(\theta_2-\theta_3)$

$$=\left[1-\frac{I_1}{g}\cdot\frac{n^2}{T_{01}}\right]-\frac{1}{T_{02}}\left\{\quad\right\},$$

and so on.

This process is continued until an expression for the total torque, which would be exerted on the shafting if it were continued after the last load, is obtained. This expression is therefore equated to zero, as no external torque can act on the system when it is in *free* vibration.

The student will probably realise that all we have done is to write down the equations of motion of the loads in turn and eliminate the θ's as we proceed.

The result is the frequency equation as before. [If the problem had been as in Fig. 26 the above process would have been continued until an expression for the amplitude at the wall (which is merely a load of infinite inertia) was obtained, and this on being equated to zero, is the frequency equation as before.]

When the number of loads is large, the solutions are most quickly obtained by assuming values for n^2 and plotting the resulting " excess torques " after the last load, hence finding the required values of n^2 to make this torque zero.

As a first approximation to the fundamental frequency the system can often be reduced to that of two loads. The number of such trial calculations will obviously depend on the experience of the operator, but care must be taken not to assume values of n^2 too far apart which would lead to a wrong interpretation of the result. The type of curve obtained is shown in Fig. 29, and if values of n^2 had been taken at points (*a*), (*b*), (*c*) an apparently continuous curve would have resulted, which would give an entirely erroneous result at *z*.

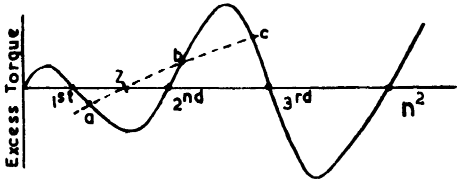

FIG. 29.

The process lends itself to most convenient tabulation, and in order to illustrate the method the following example (Fig. 30) has been worked out, the value of n^2 assumed being the actual value for the 1-noded mode of vibration, so that the final torque should be zero, the trial tables to obtain this figure not being reproduced.

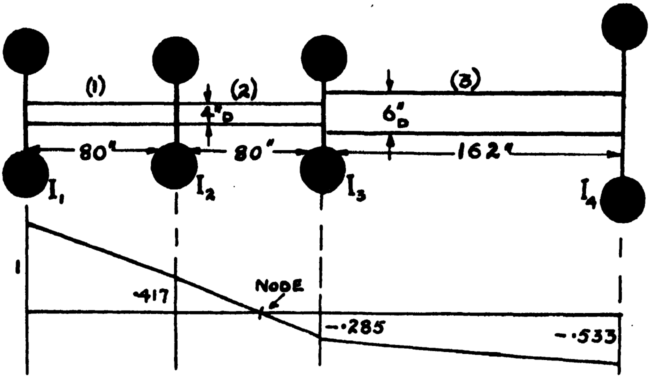

FIG. 30.

$$I_1/g = 3{,}650\ \frac{\text{lb.}}{g}\ \text{in.}^2$$

$$I_2/g = 1{,}825\quad ,,$$

$$I_3/g \doteq 1{,}825\quad ,,$$

$$I_4/g = 7{,}300\quad ,,$$

C for all shaft sections $= 10^7$ lb./sq. in.

Hence

$$T_{01} = \frac{CJ_1}{L_1} = 10^6\pi \text{ lb. in. per radian,}$$

$$T_{02} = \frac{CJ_2}{L_2} = 10^6\pi \quad ,, \quad ,,$$

$$T_{03} = \frac{CJ_3}{L_3} = 10^7 \cdot \frac{\pi}{4} \quad ,, \quad ,,$$

VALUE OF n^2 ASSUMED = 502.

Load	I/g	$\frac{I}{g}\cdot n^2$	*Load Amplitude* θ	$\frac{I}{g}\cdot n^2\cdot\theta$	$\Sigma\frac{I}{g}n^2\cdot\theta$	T_0	$\frac{\Sigma\frac{I}{g}n^2\theta}{T_0}$
1	3650	$1{\cdot}83\times10^6$	1 $-{\cdot}583=$	$1{\cdot}83\times10^6$	$1{\cdot}83\times10^6$	$10^6\pi$	·583
2	1825	$\cdot915\times10^6$	·417 $-{\cdot}702=$	$\cdot382\times10^6$ →	$2{\cdot}212\times10^6$	$10^6\pi$	·702
3	1825	$\cdot915\times10^6$	$-{\cdot}285$ $-{\cdot}2485=$	$-{\cdot}261\times10^6$	$1{\cdot}951\times10^6$	$\frac{10^7\pi}{4}$	2485
4	7300	$3{\cdot}66\times10^6$	$-{\cdot}5335$	$-1{\cdot}951\times10^6$	0	—	—
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
			↑	↓ Inertia torque for each load.	↓ Total inertia torque for all loads up to and including the one considered = Torque on shaft following.	(6) / (7) gives twist in shaft considered. This, subtracted from angular displacement of previous load, gives the amplitude of load following in column (4).	↓

The process should be self-explanatory—as we move along line (1) of the table we are merely following out the sequence of events recorded above for the calculation of the excess torque.

Since this is the final correct value of n^2 which makes the excess torque zero [column (6)], (which has been obtained after a series of similar trial tables), column (4) gives the relative amplitudes of the loads during this mode of free vibration which therefore enables the normal elastic curve to be drawn.

This is shown in Fig. 30*b*, there being one node situated in shaft section (2).

The student might like to show similarly that $n^2 = 3{,}395$ is the correct value for the two-noded mode of vibration, the nodes occurring in shafts (1) and (3).

Approximate formula for obtaining the fundamental frequency of natural torsional oscillation of a *uniform* shaft which is free to revolve and carries concentrated loads. Consider again the two-load system shown in Fig. 31*a*. We saw that the frequency equation was

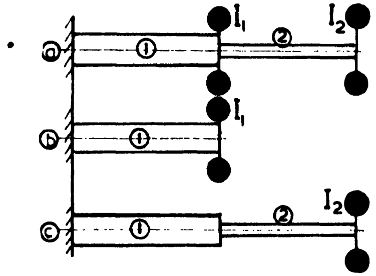

FIG. 31.

$$\frac{I_1}{g} \cdot \frac{I_2}{g}(n^2)^2 - n^2\left[\frac{I_1}{g}T_{02} + \frac{I_2}{g}(T_{01} + T_{02})\right] + T_{01}T_{02} = 0$$

(the loads and shaft sections being numbered consecutively from the wall), and from page 69 we have that, if n_1^2, n_2^2 are the roots of this quadratic equation,

$$\frac{1}{n_1^2} + \frac{1}{n_2^2} = \frac{\frac{I_1}{g}T_{02} + \frac{I_2}{g}(T_{01} + T_{02})}{T_{01}T_{02}}$$
$$= \frac{I_1}{gT_{01}} + \frac{I_2}{g}\left[\frac{1}{T_{01}} + \frac{1}{T_{02}}\right].$$

This is a special case, which can easily be proved being a quadratic, of the general result that for any number of loads on a shaft fixed at one end

$$\frac{1}{n_1^2} + \frac{1}{n_2^2} + \frac{1}{n_3^2} + \ldots = \frac{I_1}{gT_{01}} + \frac{I_2}{g}\left[\frac{1}{T_{01}} + \frac{1}{T_{02}}\right]$$
$$+ \frac{I_3}{g}\left[\frac{1}{T_{01}} + \frac{1}{T_{02}} + \frac{1}{T_{03}}\right] + \text{etc.}$$

[As we shall be dealing with a very similar result in a later chapter, the general proof is here given for reference.

Consider the equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} + \dots + a_n = 0. \qquad [\mathrm{A}]$$

This may be written as

$$a_0(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)\dots = 0,$$

where $\alpha, \beta, \gamma \dots$ are the roots of the equation, i.e.

$$a_0[x^n - (\alpha+\beta+\gamma+\dots)x^{n-1} + (\alpha\beta+\alpha\gamma+\beta\gamma+\dots)x^{n-2} \\ - (\alpha\beta\gamma+\alpha\beta\delta+\beta\gamma\delta+\dots)x^{n-3}\dots] = 0.$$

Equating the coefficients of this equation and [A] we see that

$$a_0(\alpha+\beta+\gamma+\dots) = -a_1,$$

or

$$\textit{the sum of the roots} = -\frac{a_1}{a_0},$$

and $$\Sigma(\text{roots two at a time}) = +\frac{a_2}{a_0}, \text{ etc.}$$

Now divide equation [A] by x^n, we have

$$a_0 + a_1 \cdot \frac{1}{x} + a_2\frac{1}{x^2} + a_3\frac{1}{x^3} + \dots + a_n\frac{1}{x^n} = 0,$$

or, if $y = \frac{1}{x}$,

$$a_n \cdot y^n + a_{n-1}y^{n-1} + \dots + a_0 = 0.$$

Comparing this with [A], we have

$$\text{Sum of roots} = -\frac{a_{n-1}}{a_n}.$$

But the sum of the roots $= \Sigma y = \Sigma\frac{1}{x} = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \dots$.

Hence in a homogeneous equation of the above type [A]:

$$\text{Sum of roots} = -\frac{a_1}{a_0}.$$

$$\text{Sum of reciprocals of roots} = -\frac{a_{n-1}}{a_n}.$$

It will be left to the student to show that the frequency equation for three loads numbered consecutively from the fixed end reduces to

$$I_1I_2I_3\left(\frac{n^2}{g}\right)^3 - \left(\frac{n^2}{g}\right)^2[I_1I_3(T_{02}+T_{01}) + I_3I_2(T_{01}+T_{02}) + I_1I_2T_{03}] \\ \frac{n^2}{g}[I_1T_{02}T_{03} + I_2(T_{01}T_{03}+T_{02}T_{03}) + I_3(T_{01}T_{02} + T_{02}T_{03} + T_{01}T_{03})] \\ -T_{01}T_{02}T_{03} = 0,$$

from which, taking the sum of the reciprocals of the roots, we obtain the result stated above.]

Now if $\frac{p_1}{2\pi}$ = the frequency of I_1 on shaft (1) only (Fig. 31*b*);

$$\therefore\ p_1^2 = \frac{T_{01}g}{I_1}$$

and if $\frac{p_2}{2\pi}$ = the frequency of I_2 on shafts (1) and (2) (Fig. 31*c*),

$$\therefore\ p_2^2 = \frac{gT_0}{I_2}\,;$$

where T_0 is the equivalent stiffness and is given by

$$\frac{1}{T_0} = \frac{1}{T_{01}} + \frac{1}{T_{02}}\,.$$

Thus, by a similar reasoning for any number of loads, we see that

$$\frac{1}{n_1^2} + \frac{1}{n_2^2} + \frac{1}{n_3^2} + \ldots \frac{1}{p_1^2} + \frac{1}{p_2^2} + \frac{1}{p_3^2} + \ldots\,.$$

If, as is usual, n_2^2, n_3^2 ... are large compared with n_1^2 their reciprocals may be neglected, so that very nearly

$$\frac{1}{n_1^2} \doteq \frac{1}{p_1^2} + \frac{1}{p_2^2} + \ldots\,.$$

Again, if the shaft is *uniform* (or one in which the actual shaft sections have been converted into equivalent lengths of uniform shaft)

$$\frac{1}{T_{01}} = \frac{L_1}{CJ}\,;$$

$$\frac{1}{T_{01}} + \frac{1}{T_{02}} = \frac{1}{CJ}\,[L_1 + L_2],\ \text{etc.};$$

$$\therefore\ \frac{1}{n_1^2} \doteq \frac{1}{CJg}\,[I_1L_1 + I_2(L_1 + L_2) + I_3(L_1 + L_2 + L_3) + \ldots].$$

Consider now a *uniform shaft free to revolve* (Fig. 32), and let the node for the fundamental frequency be as shown, while L_1, L_2 ... have the meanings indicated. Then the node divides the system into two cases as above and, equating the two approximate expressions for $\frac{1}{n_1^2}$, we have

$$I_1(L_1 + L_2 + L_3) + I_2(L_2 + L_3) + I_3L_3 = I_4L_4 + I_5(L_4 + L_5),$$

i.e. the node is situated at the Centre of Gravity of the loads if I_1, I_2 ... *be imagined to be masses.*

Thus approximately

$$\frac{1}{n_1^2}=\frac{1}{CJg}[I_1l_1+I_2l_2+I_3l_3+\ldots],$$

where l_1, l_2 ... are the distances of the loads *on one side* of the node from that point;

$$\therefore \text{ the fundamental frequency}=\frac{n_1}{2\pi}=\frac{1}{2\pi}\sqrt{\frac{CJg}{\Sigma Il}}.$$

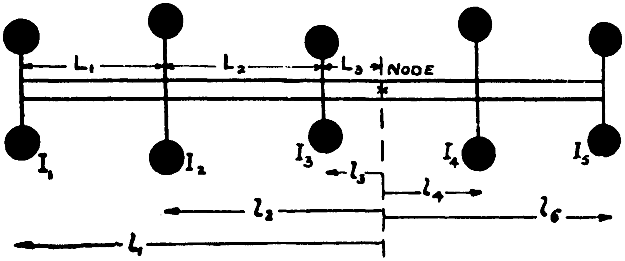

FIG. 32.

This affords by far the readiest solution for the 1-noded mode and its value from a practical point of view cannot be overestimated, since the fundamental vibration is in general the most important.

Let us work out the example for three loads given on page 73 by the various methods explained above. Fig. 33*a* shows the actual system, and Fig. 33*b* gives the equivalent system for a uniform shaft 3 in. in diameter, since:

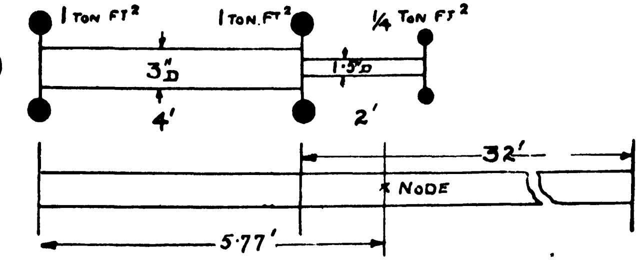

FIG. 33.

Actual shaft $=2$ ft., $1{\cdot}5$ in. diam.

$$\therefore \text{ Equivalent length of 3 in. shaft}=2\times\left[\frac{3}{1{\cdot}5}\right]^4$$

$$=32 \text{ ft.}$$

(1) *Approximate solution for the* 1-*node mode*—the node is at the C.G. of the loads if their moments of inertia are considered as " masses ".

Thus if X = the distance of the C.G. from the left-hand end, taking moments about this end, we have

$$2\tfrac{1}{4} \times X = (1 \times 4) + (\tfrac{1}{4} \times 36);$$

$$\therefore\ X = \frac{13}{2{\cdot}25} = 5{\cdot}78 \text{ ft.}$$

Thus the node is 5·78 ft. from I_1, or is situated in the second section at distance 30·22 ft. from I_3. The frequency is therefore obtained by substituting in the approximate formula above $\Sigma Il = (I_3 \times 30{\cdot}22)$, taking the loading on the right-hand side.

$$\therefore\ \text{frequency} = \frac{1}{2\pi}\sqrt{\frac{5{,}000 \times 144 \times \pi(1/4)^4 \times 32}{1/4 \times 30{\cdot}22 \times 32}}$$

$$= 5{\cdot}45 \text{ per sec.}$$

Previous calculation gave 5·50 per sec., hence the error by this approximate method is 0·91 per cent.

(2) Using the equations of motion—

We have, on substitution for $\ddot{\theta}_1$, $\ddot{\theta}_2$ and $\ddot{\theta}_3$,

$$\frac{I_1}{g} n^2\theta_1 = T_{01}(\theta_1 - \theta_2), \quad \text{......(1)}$$

$$\frac{I_2}{g} n^2\theta_2 = T_{01}(\theta_2 - \theta_1) + T_{02}(\theta_2 - \theta_3), \quad \text{......(2)}$$

$$\frac{I_3}{g} n^2\theta_3 = T_{02}(\theta_3 - \theta_2). \quad \text{......(3)}$$

Whence, on eliminating θ_1, θ_2 and θ_3, we obtain the following equation in $(n^2)^2$:

$$I_1I_2I_3\left(\frac{n^2}{g}\right)^2 - \frac{n^2}{g}[I_1I_2T_{02} + I_2I_3T_{01} + I_1I_3(T_{01} + T_{02})]$$
$$+ T_{01}T_{02}(I_1 + I_2 + I_3) = 0.$$

$$\text{Here } T_{01} = \frac{CJ_1}{L_1} = \frac{5{,}000 \times 144 \times \pi(\frac{1}{4})^4}{4 \times 32}$$

$$= 69{\cdot}10 \text{ ton ft. per radian};$$

$$T_{02} = \frac{5{,}000 \times 144 \times \pi(\frac{1}{8})^4}{2 \times 32}$$

$$= 8{\cdot}64 \text{ ton ft. per radian.}$$

Thus, on substitution, we finally arrive at

$$\left(\frac{n^2}{g}\right)^2 - 181{\cdot}2\frac{n^2}{g} + 5{,}370 = 0\,;$$

$$\therefore\ \frac{n^2}{g} = 143{\cdot}75 \text{ or } 37{\cdot}45.$$

$\therefore$ the frequencies are

$$(1)\ \frac{\sqrt{143{\cdot}75 \times 32}}{2\pi} = 10{\cdot}80\,;$$

$$(2)\ \frac{\sqrt{37{\cdot}45 \times 32}}{2\pi} = 5{\cdot}50,$$

as before.

(3) Using the method of Professor Lewis for obtaining the frequency equation, and working through from load (1) as described, the following expression for the total torque transmitted will be obtained and therefore equated to zero :

Torque due to load (1) $= \dfrac{I_1 n^2}{g}$.

Torque due to load (2) $= \dfrac{I_2 n^2}{g}\left[1 - \dfrac{I_1 n^2}{gT_{01}}\right]$.

Torque due to load (3) =

$$\frac{I_3 n^2}{g}\left\{\left[1 - \frac{I_1 n^2}{gT_{01}}\right] - \frac{1}{T_{02}}\left[\frac{I_1 n^2}{g} + \frac{I_2 n^2}{g}\left(1 - \frac{I_1 n^2}{gT_{01}}\right)\right]\right\}.$$

Adding these and equating to zero, we obtain

$$\left(\frac{n^2}{g}\right)^2\left[\frac{I_1 I_2 I_3}{T_{01} T_{02}}\right] - \frac{n^2}{g}\left[\frac{I_1 I_2}{T_{01}} + I_1 I_3\left(\frac{1}{T_{01}} + \frac{1}{T_{02}}\right) + \frac{I_2 I_3}{T_{02}}\right]$$
$$+ (I_1 + I_2 + I_3) = 0,$$

which, as would be expected, is the same as the equation just deduced above.

Alternative method for the determination of the frequency equation. There are many variations of the method for determining the frequency equation (or its trial and error solution) by working along the shaft, and mention will here be made of one other since this method has been employed for the solution of a special problem in a later chapter.

Consider again the general system of a shaft which is free to revolve and which carries a series of flywheel loads (Fig. 34) :

The torque T_1, exerted on load (1) when the angular displacement is θ_1, is

$$T_1 = \frac{I_1}{g}\frac{d^2\theta_1}{dt^2} = -\frac{I_1}{g}n^2\theta_1,$$

or

$$\frac{T_1}{\theta_1} = -\frac{I_1}{g}n^2.$$

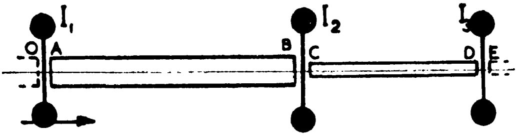

FIG. 34.

Since this represents the torque exerted on the load when *at unit angular displacement* during vibration, the term *Dynamic Stiffness*, denoted by the symbol Z, has been suggested for it, being analogous to the static stiffness T_0 of a shaft in torsion.

Consider now the shaft section AB. As it possesses no inertia (our original assumption), any torque T applied to it will be transmitted along the shaft, so that in transferring from A to B

$$Z_A = \frac{T}{\theta_1}, \qquad Z_B = \frac{T}{\theta_2},$$

and as $(\theta_2 - \theta_1) = \frac{T}{T_{01}}$ with the datum at B, we see that

$$\frac{1}{Z_B} - \frac{1}{Z_A} = \frac{1}{T_{01}},$$

or

$$\frac{1}{Z_B} = \frac{1}{Z_A} + \frac{1}{T_{01}}.$$

Let us work along the shaft from left to right starting at the point O just before load (1) where Z_0 will be zero.

Passing load (1), $Z_A - Z_0 = -\frac{I_1}{g}n^2,$

i.e.

$$Z_A = -\frac{I_1}{g}n^2.$$

Transferring to B along shaft AB,

$$\frac{1}{Z_B}=\frac{1}{Z_A}+\frac{1}{T_{01}}=\frac{1}{-\frac{I_1}{g}n^2}+\frac{1}{T_{01}};$$

$$\therefore\ Z_B=\frac{\frac{I_1}{g}n^2\,.\,T_{01}}{\frac{I_1}{g}n^2-T_{01}}.$$

Passing load (2), $Z_C-Z_B=-\frac{I_2}{g}n^2$;

$$\therefore\ Z_C=\frac{\frac{I_1}{g}n^2\,.\,T_{01}}{\frac{I_1}{g}n^2-T_{01}}-\frac{I_2}{g}n^2$$

$$=\frac{\frac{I_1}{g}n^2T_{01}-\frac{I_2}{g}n^2\left(\frac{I_1}{g}n^2-T_{01}\right)}{\left(\frac{I_1}{g}n^2-T_{01}\right)}.$$

If there was no continuation after this load, i.e. the system consisted of the two loads I_1, I_2, at the ends of a shaft, Z_C would be equated to zero, i.e.

$$0=\frac{I_1}{g}n^2T_{01}-\frac{I_1}{g}\frac{I_2}{g}(n^2)^2+\frac{I_2}{g}n^2T_{01},$$

or

$$n^2=gT_{01}\left[\frac{I_1+I_2}{I_1I_2}\right],$$

as we have obtained by other methods before.

Carrying on for the three-load system depicted, we have

Shaft CD, $$\frac{1}{Z_D}=\frac{1}{Z_C}+\frac{1}{T_{02}},$$

and finally

Passing load (3) $$Z_E-Z_D=-\frac{I_3}{g}n^2,$$

and as Z_E is here zero,

$$Z_D=+\frac{I_3}{g}n^2,$$

which is merely another way of saying that the Z-value just to the left of point D is equal and opposite to the value just to the right.

It will be left to the student to make the final substitutions and so obtain the same frequency equation as before. It may sometimes be convenient to work from both ends of the shaft as similar expressions will be formed, and at the meeting point the Z values so obtained must be equal and opposite.

Many methods of expressing the frequency equation have been evolved by different investigators, and in problem 16, Examples I, one other form is shown.

Geared drives. The natural frequencies of torsional oscillation of any shaft arrangement, coupled by gearing, can readily be obtained, *provided that we make one important assumption, namely that contact never ceases between the teeth in gear, the teeth being pressed together firmly by the transmitted torque.* In other words, the torque at a gear face must never become negative, since, if the teeth part, the system is no longer continuous.

There are various methods of dealing with the problem. The equations of motion of the loads can be written down in the usual way, and the reaction torques of the gear wheel on its pinion and that of the pinion on its wheel included in the respective equations for the pinion and the wheel, e.g. Fig. 35 :

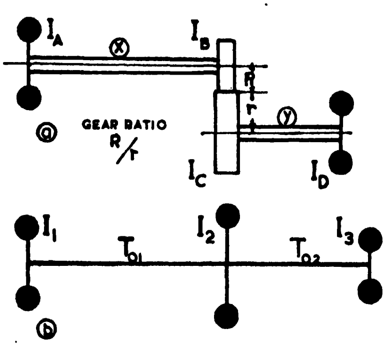

Fig. 35.

When gear pinion I_B is at an angular displacement θ_B, measured positive, say clockwise, the restoring torque due to shaft x is $-T_{0x}(\theta_B-\theta_A)$ anticlockwise. But besides this torque there is another torque acting due to the reaction of the driven wheel I_C on I_B, and its direction is anticlockwise for θ_B clockwise. The magnitude of this torque is PR, where P is the tangential reaction between the gear wheels, so that the equation of motion will be

$$\frac{I_B}{g}\ddot{\theta}_B = -T_{0x}(\theta_B - \theta_A) - PR.$$

On transference to the next shaft which rotates in the opposite direction to the previous one, instead of considering negative angles, let us reverse the direction for measuring the angular displacements, anticlockwise now being +ve; so that, for a clockwise rotation θ_B, I_C rotates through an angle $\theta_C = (R/r)\theta_B$ anticlockwise and the reaction P, being now that of I_B on I_C, produces a torque Pr in the direction of the displacement, i.e. anticlockwise, and therefore acts in the opposite direction to the restoring torque of the shaft (y) which is $-T_{0y}(\theta_C - \theta_D)$ clockwise. Thus,

$$\frac{I_C}{g}\ddot{\theta}_C = -T_{0y}(\theta_C - \theta_D) + Pr.$$

Since θ_C is known in terms of θ_B, this reduces the variables by one, and sufficient equations will be found to eliminate the θ's *and* P, leaving the frequency equation as usual.

Again, if using one of the methods of working along the shaft, when we arrive at the expressions for the torque T_P on the gear pinion with the corresponding angular displacement θ_P, this torque will be divided by the gear ratio R/r (see Fig. 35) while the angular displacement will be multiplied by R/r to obtain the corresponding values when transferring across from the pinion to its wheel.

By far the best method, however, is to replace the actual system by an equivalent system of continuous shaft so that all loads and shaft sections rotate at the same speed, the gearing being eliminated. We have, therefore, to obtain expressions for the equivalent inertia of a load and the equivalent stiffness of a shaft section so that the vibrational characteristics will be unaltered, i.e. the kinetic energy of a load, and the strain energy in, and power transmitted through, a shaft section must be the same in both cases.

Let N_A = the actual r.p.m. of the load or shaft section concerned.

N_E = the equivalent „ „ „ „ „ „

For a load: $\text{K.E.} = \frac{1}{2}\frac{I_A}{g}\omega_A^2 = \frac{1}{2}\frac{I_E}{g}\omega_E^2$;

$$\therefore\ I_E = I_A\left[\frac{\omega_A}{\omega_E}\right]^2 = I_A\left[\frac{N_A}{N_E}\right]^2.$$

For a shaft section:

If T_A = the actual torque transmitted by the section, then for the same power

$$T_E\omega_E = T_A\omega_A;$$

$$\therefore\ T_E = T_A\cdot\frac{\omega_A}{\omega_E} = T_A\cdot\frac{N_A}{N_E},$$

and, for the same strain energy,

$$\tfrac{1}{2}T_A\theta_A = \tfrac{1}{2}T_E\theta_E \quad \text{(see page 46)}$$

so that for the equivalent torsional stiffness T_{0E}, we have

$$\frac{T_{0E}}{T_{0A}} = \frac{T_E/\theta_E}{T_A/\theta_A} = \frac{T_E}{T_A}\cdot\frac{\theta_A}{\theta_E} = \left(\frac{N_A}{N_E}\right)^2.$$

Thus the equivalent inertia of a load or stiffness of a shaft is obtained by multiplying the actual by the square of the ratio of the actual to equivalent speeds. It is therefore preferable to reduce a system to the common speed of the high speed shaft as the equivalent values will be less unwieldy, being smaller.

Let us take some examples:

(1) Fig. 35*a*, which might represent an electric motor drive through a simple gear to a pump runner.

Replacing this actual system by a dynamically equivalent system in which all loads and shaft sections rotate at the speed (say) of shaft *x*, we have (Fig. 35*b*):

$$I_1 = I_A, \qquad T_{01} = T_{0x},$$

$$I_2 = \left[I_B + I_C\left(\frac{N_y}{N_x}\right)^2\right],$$

$$I_3 = I_D\left(\frac{N_y}{N_x}\right)^2, \qquad T_{02} = T_{0y}\left(\frac{N_y}{N_x}\right)^2,$$

so that the system reduces to the simple case of three loads.

(2) Fig. 36*a*, which might represent two turbine rotors driving through gearing a propeller I_A—or an engine I_A driving some twin system.

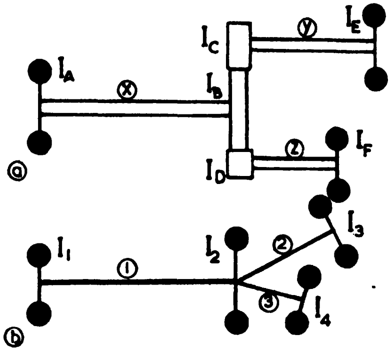

FIG. 36.

Fig. 36*b* represents diagrammatically the equivalent system of four loads, and, reducing everything to the speed of shaft x,

$$I_1 = I_A,$$

$$I_2 = I_B + I_C\left(\frac{N_y}{N_x}\right)^2 + I_D\left(\frac{N_z}{N_x}\right)^2, \qquad T_{01} = T_{0x}$$

$$I_3 = I_E\left(\frac{N_y}{N_x}\right)^2, \qquad T_{02} = T_{0y}\left(\frac{N_y}{N_x}\right)^2,$$

$$I_4 = I_F\left(\frac{N_z}{N_x}\right)^2, \qquad T_{03} = T_{0z}\left(\frac{N_z}{N_x}\right)^2.$$

In this replacement a double branch appears in the equivalent system, but the equations of motions are written down as usual, e.g.

$$(1)\quad \frac{I_1}{g}n^2\theta_1 = T_{01}(\theta_1 - \theta_2);$$

$$(2)\quad \frac{I_2}{g}n^2\theta_2 = T_{01}(\theta_2 - \theta_1) + T_{02}(\theta_2 - \theta_3) + T_{03}(\theta_2 - \theta_4);$$

$$(3)\quad \frac{I_3}{g}n^2\theta_3 = T_{02}(\theta_3 - \theta_2);$$

$$(4)\quad \frac{I_4}{g}n^2\theta_4 = T_{03}(\theta_4 - \theta_2).$$

Upon eliminating the θ's a cubic in $(n^2)^3$ will result, and there will be one, two, or three nodes.

An interesting and important case of a symmetrical arrangement of the above system is illustrated in Example 32 (page 118). In many cases the inertia of the gearing will be negligible in comparison with that of the other loads, so that the problem can be still further simplified.

If we had been working by the tabular method, when we pass the gear pinion I_B we should have to form two lines in the table, one for each branch, and the necessary requirement is that the excess torque must be zero at the end of each branch.

A full discussion of every type of problem connected with shaft systems is given in the book *Practical Solution of Torsional Vibration Problems* by Dr. W. Ker Wilson.*

Gearing in angular motion is analogous to a lever, and we should be able to get corresponding expressions for the equivalent mass of a load or the stiffness of a spring when transferred from their actual positions on a lever system to some other position more convenient for calculation purposes. Consider a rod, pivoted at a point O, the angular motion of which is controlled by a spring of stiffness F_A attached at a point A distant L_A from O. The other end of the spring is fixed and the spring is arranged at right angles to the rod when at rest.

If we wish to replace this actual spring by another of stiffness F_E at a distance L_E from O, the requirement will be that for a given small angular displacement θ of the rod, the equivalent spring must exert the *same restoring torque*.

Thus,

$$\text{deflexion of spring } A = L_A\theta,$$

$$\text{,,} \quad \text{,,} \quad \text{,,} \quad E = L_E\theta;$$

$$\therefore\ F_A(L_A\theta) \,.\, L_A = F_E(L_E\theta) \,.\, L_E,$$

or

$$F_E = F_A \,.\, \left(\frac{L_A}{L_E}\right)^2.$$

Similarly, an equivalent mass m_E at a point E must have the *same moment of inertia* about O as the actual mass m_A at the point A, in order to give the same effect, i.e.

$$m_E = m_A \left(\frac{L_A}{L_E}\right)^2.$$

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* Chapman and Hall.

Torsional oscillations of a crankshaft.

(*a*) *To take into account the reciprocating masses.* In our previous discussions on shaft systems in torsional oscillation, we have assumed that all the attached masses were purely revolving in character having constant moments of inertia, but with an engine crankshaft, which is one of the most important systems where trouble can arise from torsional oscillation, there are reciprocating masses to complicate the problem.

We can gain a visual idea of the effect of such a mass as follows. A reciprocating mass of weight W, its connecting rod and crank, are shown in Fig. 37, and when the crank is at either dead centre it will be realised that oscillations of the crankshaft can occur without appreciably being affected by the reciprocating mass, the movement of the latter being very small for a considerable angular movement of the crank.

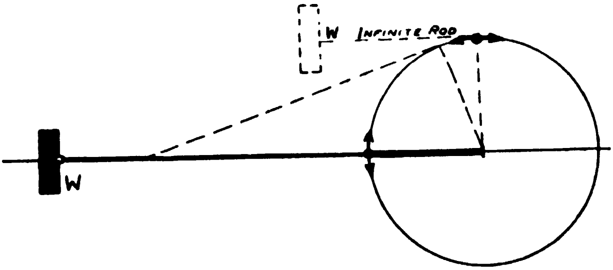

FIG. 37.

When, however, the crank and connecting rod are about at right angles—or, considering an *infinite* rod, when the crank is at right angles to the line of stroke, any vibration of the shaft is transmitted to the full mass, just as if the whole mass were concentrated at the crankpin. Thus, with an infinite rod, the effect of the reciprocating mass is equivalent to that of a rotating mass, the weight of which varies from zero to W every quarter revolution, concentrated at crank radius R.

Instead of this variable system, it is usual to assume a constant mass of weight $\frac{1}{2}W$ concentrated at the crankpin, so that its equivalent moment of inertia will be $\frac{1}{2}WR^2$ absolute units. This is the best we can do, but, in any case, to reduce an actual shaft with flanges, couplings, etc., to a form suitable for mathematical treatment, we must introduce various approximations.

We can examine the problem more mathematically as follows:

To determine the moment of inertia of an equivalent disk, which will cause the same torsional vibrations of a shaft system as a reciprocating mass.

During a free vibration we know that

$$\text{K.E.} + \text{P.E.} = \text{constant}.$$

The potential energy is strain energy of angular twist of the shaft, and will be the same in the actual and equivalent conditions, as the shaft system is unaltered.

Thus we need only consider the kinetic energy.

Let the mechanism be set at any given crank angle θ (measured from the inner dead centre), the shaft *not* being in rotation, and let the shaft be set into angular oscillation about this position as mean, causing the crank to have an angular velocity ω (say) as it passes through this position.

Assuming an infinite connecting rod, the reciprocating parts will move with simple harmonic motion and their velocity at this given instant will be

$$-\omega R \sin \theta,$$

therefore, if the weight of the reciprocating masses $= W$, the kinetic energy at this instant is

$$\frac{1}{2}\frac{W}{g}\omega^2 R^2 \sin^2 \theta,$$

so that the moment of inertia I of an equivalent disk, which will have the same K.E. at the same angular velocity, will be given by

$$\frac{1}{2}\frac{I}{g}\omega^2 = \frac{1}{2}\frac{W}{g}\omega^2 R^2 \sin^2 \theta,$$

i.e. $$I = WR^2 \sin^2 \theta \text{ absolute units}.$$

The value of I thus varies with the crank position, and if the shafting were at rest with the crank set at this angle θ, the frequency of vibration would be calculated with the above value of I.

However, we are more concerned with a revolving shaft which is undergoing vibration, and we can approximate to the result by taking the mean value of I during one revolution (i.e. $\theta = 0 \rightarrow 2\pi$ radians).

Thus,

$$\text{Mean } I = \frac{WR^2 \int_0^{2\pi} \sin^2 \theta \, d\theta}{2\pi}.$$

Now $\int \sin^2 \theta \, d\theta = \frac{\theta}{2} - \frac{\sin 2\theta}{4}$, but as we are integrating between 0 and 2π the sine term vanishes, so that

$$\text{Mean } I = \frac{1}{2\pi} WR^2 \cdot \frac{2\pi}{2} = \frac{WR^2}{2},$$

which is the same as the moment of inertia of a mass of weight $\frac{W}{2}$ concentrated at crank radius. This is the usual value assumed in practice, but if a nearer approximation is needed, the expression for the velocity of the reciprocating parts with a finite rod (see page 215) could be used, the mean value of I then being

$$\frac{W}{2}\left[1 + \frac{1}{4}\left(\frac{R}{L}\right)^2\right] \cdot R^2,$$

where L = the length of the connecting rod.

(*b*) *Elasticity of a crankshaft.* The problem of estimating accurately the torsional elasticity of such a complicated structure as a crankshaft is extremely difficult. An applied torque will produce twisting of the shaft and of the crankpins as well as bending of the webs, while the amount of restraint exercised by the bearings will influence matters. Again this restraint will depend on the clearance and on the thickness of the oil film.

Thus it will be seen that it is no easy matter to derive a simple formula for replacing the cranks by an equivalent length of uniform shaft. With the usual crank design in which the diameter of the pin is equal to that of the shaft, it will be clear that the smaller the throw (crank radius) in relation to the diameter of the shaft, the more nearly will the system approach that of a uniform shaft, while the longer the throw, the greater will be the angle of twist produced by a given torque due to the increased elasticity of the webs.

A most interesting article on this subject by Major B. C. Carter is to be found in *Engineering*, July 13th 1928, and a further article on the same subject appeared on November 1st 1929, while a great wealth of practical information will be found in Dr. Ker Wilson's book previously mentioned.*

If it is possible to apply a torque at one end of an actual shaft, the other end being rigidly held, the torsional stiffness can be determined directly, but as these calculations may have to be made during the drawing-board stage of a design no experimental approach can then be made.

Combined linear and angular oscillations. As another important example of the use of the equations of motion, let us take the case of a load supported on two springs and free to vibrate in a vertical plane, but without lateral movement (Fig. 38*a*).

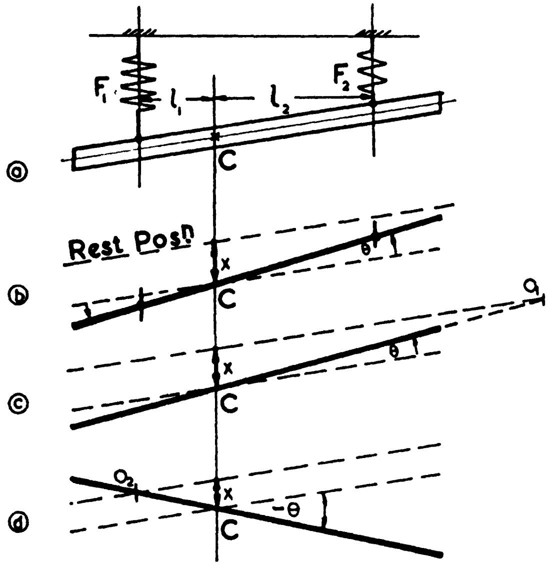

FIG. 38.

While the centre of gravity C of the load undergoes a linear oscillation, the load itself will, in general, perform an angular oscillation about the C.G., and since the configuration at any instant can be expressed by the linear displacement x of the C.G. and the angle θ turned through by the load from its rest position, the system has therefore two degrees of freedom and will possess two natural modes of vibration.

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* Ibid., p. 96.

Let W = the wt. of the body;

$I = Wk^2$ = the moment of inertia (abs. units) of the body, about an axis through its C.G. and perpendicular to the plane of vibration.

F_1, F_2 = the spring stiffnesses.

Then, assuming conditions to be as in Fig. 38*b*, at the instant shown the extension of spring (1) from its rest position is $(x + l_1\theta)$, and that of spring (2) is $(x - l_2\theta)$, x and θ being considered of the same sign when the load is moving down with anti-clockwise rotation, or their reverse.

Now the effect of each spring force in regard to the C.G. is to introduce (1) at C an equal and parallel force, which will give the linear acceleration and (2) a couple about C producing angular acceleration.

Thus the equation of linear motion is

$$\frac{W}{g}\frac{d^2x}{dt^2} = -F_1(x + l_1\theta) - F_2(x - l_2\theta).$$

As regards the angular motion, spring (1) is exerting a clockwise torque $F_1(x + l_1\theta) \, . \, l_1$ and spring (2) an anti-clockwise torque $F_2(x - l_2\theta)l_2$. The diagram shows x greater than $l_2\theta$; if less, the torque changes sign, the spring becoming compressed.

Hence for θ positive anti-clockwise

$$\frac{I}{g}\frac{d^2\theta}{dt^2} = -F_1(x + l_1\theta)l_1 + F_2(x - l_2\theta) \, . \, l_2.$$

Thus

$$\frac{W}{g}\frac{d^2x}{dt^2} = -(F_1 + F_2)x - (F_1l_1 - F_2l_2)\theta;$$

$$\frac{I}{g}\frac{d^2\theta}{dt^2} = -(F_1l_1 - F_2l_2)x - (F_1l_1^2 + F_2l_2^2)\theta;$$

or

$$\frac{d^2x}{dt^2} + ax + b\theta = 0, \qquad \text{........................(1)}$$

$$\frac{d^2\theta}{dt^2} + \frac{b}{k^2}x + \frac{c}{k^2}\theta = 0,$$

i.e.

$$\frac{d^2\theta}{dt^2} + dx + f\theta = 0, \qquad \text{........................(2)}$$

where $$a = -(F_1 + F_2)\frac{g}{W}, \qquad b = -(F_1 l_1 - F_2 l_2)\frac{g}{W},$$

$$c = -(F_1 l_1^2 + F_2 l_2^2)\frac{g}{W}, \qquad d = \frac{b}{k^2}, \qquad f = \frac{c}{k^2}.$$

Now the above two equations normally contain both x and θ indicating that both linear and angular vibrations occur in each natural mode, but if $b=0$, it will be seen that x and θ become independent, equation (1) representing entirely linear motion, and equation (2) angular motion only. The solutions of these equations give the two normal modes, the one being a linear vibration without rotation, the load being at every instant parallel to its rest position, the other being an angular oscillation about the C.G. which now suffers no linear displacement. (These normal modes can, of course, be superimposed so that a combined linear and angular motion could occur.)

Let us consider the general case in which $b \neq 0$ so that x and θ are not independent.

To solve, assume

$$x = A \cos(nt + e) \text{ so that } \frac{d^2x}{dt^2} = -n^2 x,$$

$$\theta = B \cos(nt + e) \quad ,, \quad ,, \quad \frac{d^2\theta}{dt^2} = -n^2\theta.$$

Thus
$$x(a - n^2) + b\theta = 0, \quad \text{........................(i)}$$
$$x \, . \, d + (f - n^2)\theta = 0. \quad \text{........................(ii)}$$

Eliminating x and θ we get

$$(a - n^2)(f - n^2) - bd = 0,$$

or
$$(n^2)^2 - n^2(f + a) + (af - bd) = 0;$$

$$\therefore \; n^2 = \tfrac{1}{2}[(f + a) \pm \sqrt{(f + a)^2 - 4(af - bd)}],$$

from which the frequencies of the two natural modes $\frac{n_1}{2\pi}, \frac{n_2}{2\pi}$ are obtained.

Substituting the values n_1, n_2 in equation (i) we get the ratio $\frac{x}{\theta}$ or $\frac{A}{B}$ for each mode, i.e.

$$\frac{x}{\theta} = -\frac{b}{a - n^2} \quad \text{or} \quad \frac{b}{n^2 \quad a}.$$

It will be found that this ratio is positive for the slower mode (n_1) and negative for the higher (n_2). Remembering that we have taken x and θ of the same sign when the linear motion is downwards and the angular anti-clockwise (or their reverse), it will be seen from Figs. 38*c* and *d* that the actual motions in each mode are the same as if the body was swinging about the points O_1, O_2 respectively, being distant from C to the right and left by the amounts CO_1, $CO_2 = x/\theta$.

If the two frequencies are very nearly equal, then, as discussed on page 58, the general motion due to the superposition of the two natural modes will result in beats. In order to visualise matters, consider the motion of a uniform bar AB symmetrically mounted on two springs of equal stiffness, the rest position being horizontal (Fig. 39). Thus, in our equations above, $b = 0$, and the one natural mode is entirely linear:

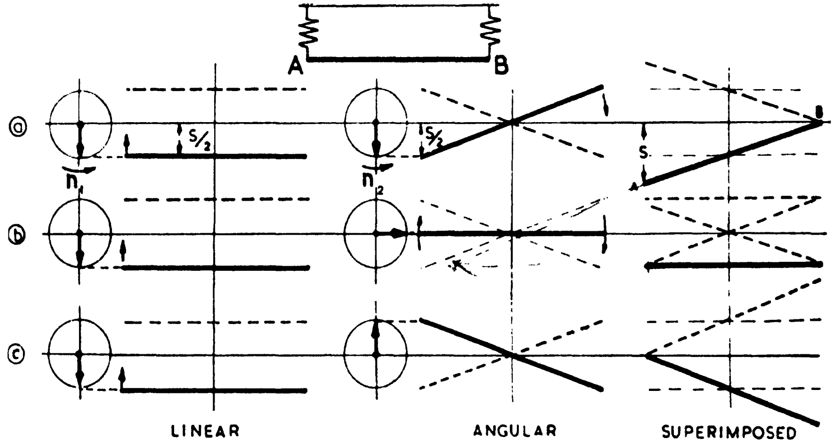

FIG. 39.

$$\frac{W}{g}\frac{d^2x}{dt^2} = -2Fx,$$

so that $x = y_1 \cos(n_1 t + e_1)$ and $n_1 = \sqrt{(2Fg/W)}$, and the other mode is entirely angular:

$$\frac{I}{g}\frac{d^2\theta}{dt^2} = -(2Fl^2)\theta,$$

so that $\theta = z_1 \cos(n_2 t + e_2)$ and $n_2 = \sqrt{\frac{2Fl^2 g}{I}}$.

Suppose that the frequency of the angular oscillations is the greater, and let the end *A* be pulled down a given distance *S* while the end *B* is held in its rest position (Fig. 39*a*). It will be seen that when these initial imposed conditions are split up into the two component motions, the amplitude of the end *A* in each motion will be *S*/2. Upon release, the system will for a short time continue to oscillate about the point *B* as a fixed point, the motion being a combination of the two natural modes which start off in phase with their maximum displacements as shown in Fig. 39*a*.

Gradually, however, the phase of the angular motion gains on the linear and Fig. 39*b* shows the result when the phase difference is 90 degrees. It will be seen that the bar is horizontal at its maximum displacement, there being no angular deflexion, while as it passes the mean position the bar suffers the full angular deflexion.

Finally, Fig. 39*c* shows the component motions 180 degrees out of phase, and it will be seen that the point *A* is now stationary, spring *B* only being in vibration.

The whole process then reverses.

By turning the diagram upside down we obtain the equivalent of a motor car body on its springs. The subject of the vibration of vehicles is discussed in the paper " The Principles of Vehicle Suspension " by H. S. Rowell.*

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* *Proc. Inst. Automobile Engineers*, 1923 (Part II).

EXAMPLES I

CHAPTERS I, II, and III

1. Determine the maximum force required to accelerate the reciprocating parts of a single cylinder engine, for which the following data apply:

Mass of reciprocating parts	500 lb.
Stroke	18 in.
R.P.M.	250.

The connecting rod may be assumed infinitely long.

Ans. 8,000 lb. wt.

2. The track of a switchback railway is arranged in a wave formation, such that the car over this section is given a simple harmonic motion in a vertical plane. If the vertical distance from crest to hollow is 20 ft. and the time taken by the car to travel from crest to crest is 4 sec., determine the maximum and minimum pressures on the rails, the weight of the car being 2,000 lb. [Liv. U.]

Ans. 3,540 lb. wt.; 460 lb. wt.

3. The two ends of a long rod are given simple harmonic motions of the same period along paths which, the rod being long, may be taken as parallel straight lines. The amplitudes of the motions are 2 in. and 6 in. respectively, while the phase of the motion with the smaller amplitude is 90° in advance of the other. Determine the amplitude and phase of the mid-point of the rod.

Ans. 3·16 in. 18° 26′ in advance of the motion with greater amplitude.

4. In a certain examination candidates were asked to find the greatest value of the acceleration of a mass oscillating at the end of an elastic string with a period of $\frac{1}{2}$ sec. and a travel of 6 in. Determine the answer required and show that no elastic string could maintain a mass in oscillation with a travel of 6 in. and with a period so small as $\frac{1}{2}$ sec. [Liv. U.]

Ans. 39·5 ft. sec.^{2}

Here

$$\text{Period} = \frac{2\pi}{n}; \quad \therefore \quad n = \frac{2\pi}{\frac{1}{2}} = 4\pi \text{ radians/sec.}$$

$$\text{S.H. acceleration} = -n^2x;$$

$$\therefore \text{ Max. acceleration} = -16\pi^2 \times \frac{3}{12} = -39{\cdot}5 \text{ ft. sec.}^2.$$

This acceleration is directed towards the rest position and when the body reaches the highest point of its oscillation the support would have to be *in compression* to give this value which is greater than gravity. With a string this is obviously impossible ; the ½ travel or amplitude with such a support cannot be greater than the static extension, the maximum acceleration being then just equal to gravity.

5. A vertical U-tube, of uniform bore throughout, contains liquid occupying a total length of 4 ft., with equal heights of column in either limb. Determine the period of the resulting oscillations when the liquid column is displaced and then released. [Liv. U.]

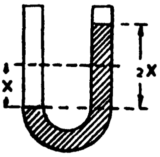

Fig. (*a*).

Let $2L$ = total length of liquid in tube. Fig. (*a*).

w = wt. per unit length.

∴ Total mass $= 2L \times w$.

When the column is depressed a distance x ft. from its rest position in one limb, the height of the column in the other limb is $2x$ ft. above that in the first limb.

Hence restoring force $= w \times 2x$;

$$\therefore \frac{2wL}{g} \cdot \frac{d^2x}{dt^2} = -2wx,$$

or

$$\frac{d^2x}{dt^2} = -\frac{g}{L}x;$$

$$\therefore \text{Period} = 2\pi\sqrt{\frac{L}{g}}.$$

Here $2L = 4$;

∴ Period = 1·57 secs.

The result is interesting since the period does not depend on the nature of the liquid but only on the total length.

6. A helical spring, when fixed at one end and loaded at the other, requires a force of 10 lb. to produce an elongation of 1 in. The ends of the spring are now rigidly fixed, the one vertically above the other and a load of 20 lb. is attached to the middle of its length. Determine the time of one complete vibration when the load is set vibrating in a vertical plane. *Ans.* 0·227 sec.

7. A load of ½ ton is being raised by means of a steel rope which weighs 1·5 lb. per foot and has a cross-sectional area of 0·5 sq. in.

At what depth will the natural frequency of longitudinal vibration of the loaded rope be 5 per sec. ? Take $E = 7{,}000$ tons per sq. in. for the steel rope. *Ans.* 207 ft.

8. The mainmast of a large vessel is of practically uniform diameter; its height above the wedging is 84 ft., and the centre of oscillation is 16 ft. below the wedging. The weight of the part above the wedging is 25 tons.

Assuming simple harmonic motion, if the period for a single roll from side to side is 6 secs., determine the bending moment on the mast near the wedging when the ship has rolled 30° from the vertical. [Liv. U.]

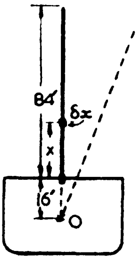

FIG. (*b*).

We have [Fig. (*b*)]

(1) B.M. due to static inclination of 30°

$$=25\times\frac{84}{2}\cdot\sin 30^\circ=525 \text{ tons ft.}$$

(2) For dynamic moment

$$d^2\theta/dt^2=-n^2\theta.$$

$$\text{Period}=12 \text{ secs.}=\frac{2\pi}{n};$$

$$\theta=30^\circ, \text{ i.e. } \pi/6 \text{ radians.}$$

Hence angular acceleration of mast at $\theta=30^\circ$

$$=-\left[\frac{2\pi}{12}\right]^2\cdot\frac{\pi}{6}=-\left[\frac{\pi}{6}\right]^3 \text{ radians/sec.}^2$$

Consider an element of mast distant x from wedging, if $w=$ wt. per unit length, then

Moment of inertia of element about centre of oscillation O

$$=\frac{w}{g}\cdot\delta x\cdot(x+16)^2;$$

$$\therefore \text{ Torque about } O \text{ at } 30^\circ=\frac{w}{g}\cdot\delta x\cdot(x+16)^2\cdot\left(\frac{\pi}{6}\right)^3;$$

$$\therefore \text{ Torque about wedging}=\frac{w}{g}\cdot\delta x\cdot(x+16)^2\cdot\left(\frac{\pi}{6}\right)^3\times\frac{x}{x+16};$$

$$\therefore \text{ Total torque}=\frac{w}{g}\left(\frac{\pi}{6}\right)^3\int_0^{84}(x+16)x\,.\,dx$$

$$=338 \text{ tons ft.}$$

Now as mast moves outwards from the centre the system is being retarded, but owing to the momentum acquired by the mast it will tend to be carried on and so exert an outward B.M. about wedging.

If mast is moving inwards the system is being accelerated, but owing to the mast's inertia it will tend to lag behind, hence the B.M. exerted at wedging is still outwards.

$$\therefore \text{ Total B.M.}=338+525=863 \text{ tons ft.}$$

9. A disc of weight 20 lb. and diameter 18 in. was arranged to undergo angular oscillations in a horizontal plane about a vertical axis passing through its centre of gravity, being controlled by the torsion of an elastic rod.

It was found that the disc made 16·5 complete oscillations per minute.

A body of weight 5 lb. was then placed on the disc and the number of vibrations per minute was now found to be 13·2.

Determine the radius of gyration of the body in the position in which it was placed about the axis of the torsional oscillations.

Ans. 0·795 ft.

10. The diagram Fig. (*c*) represents a light rigid rod of negligible mass, free to turn about the fixed point O and carrying at its other end a concentrated load of 2 lb. wt.

A spring of stiffness 4·8 lb. per inch extension is attached to the mid-point of the rod as shown.

Determine the period of the oscillations set up when the load is given a small displacement from its rest position.

Ans. 0·414 sec.

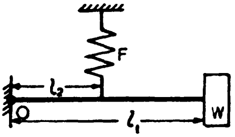

Fig. (*c*)

A common error in this problem is to say that the equivalent load on spring if attached to its end would be $2W$, being twice the weight at half the distance. This, however, is not the case, since the load is undergoing *angular oscillations* about O, and hence W at radius of gyration l_1 is equivalent to $W \times [l_1/l_2]^2$ at a radius of gyration l_2, i.e. $4W$ in this case (see page 96).

Hence equivalent load at spring in linear motion $= 4W$.

$$\therefore \text{ Period} = 2\pi\sqrt{\frac{4W}{Fg}}$$

$$= 0{\cdot}414 \text{ sec.}$$

Alternatively, working from first principles,

let $x =$ extension of spring, when

$\theta =$ angle turned through by rod ;

then, restoring torque $= Fx \,.\, l_2$

$$= F \,.\, \theta \,.\, l_2^2, \text{ since } x = \theta \,.\, l_2 ;$$

$$\therefore \quad -F \,.\, \theta \,.\, l_2^2 = \frac{W}{g} l_1^2 \,.\, d^2\theta/dt^2,$$

or
$$d^2\theta/dt^2 = -\frac{Fg}{W\,[l_1/l_2]^2} \,.\, \theta.$$

Hence period as above.

11. A weight W lb. hangs from a cord attached to the rim of a disk of radius R ft., as shown in Fig. (*d*). The disk is free to turn about a horizontal axis through its centre O, and is restrained by a spring attached at radius r ft., the rest position being as indicated.

Determine the period of the oscillations resulting from a small displacement of the weight.

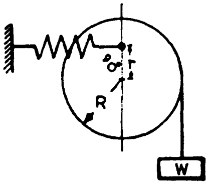

FIG. (*d*).

Moment of inertia of disk $= I$ lb. ft.^{2}

Stiffness of spring $= F$ lb. per ft.

[Liv. U.]

Ans. $\text{Period} = 2\pi\sqrt{\dfrac{I + WR^2}{gFr^2}}$.

12. A uniform cylinder, closed at the base, is weighted so as to float upright in a given liquid, the depth immersed being 1 ft.

Neglecting any resistance offered by the liquid to the motion, determine the frequency of the oscillations that will be set up when the system is given a vertical displacement from its rest position and released. *Ans.* 0·904 per sec.

13. A uniform beam of weight W rests horizontally on two rough rollers rotating at high speed in opposite directions as shown in Fig. (*e*) i. Show that, if the beam moves at all, it will move with simple harmonic motion along its axis. If μ_1 and μ_2 are the coefficients of friction between the beam and the rollers, obtain an expression for the period and for the distance of the centre of the oscillations from the centre of symmetry shown in the figure.

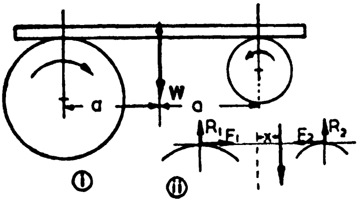

FIG. (*e*).

Let the beam be displaced a distance x to the right (Fig. (*e*) ii). Then the normal reactions of the rollers are

$$R_1 = W \cdot \frac{a-x}{2a}, \qquad R_2 = W \cdot \frac{a+x}{2a};$$

and therefore the frictional forces are

$$F_1 = \mu_1 \cdot W \cdot \frac{(a-x)}{2a}, \qquad F_2 = \mu_2 W \cdot \frac{(a+x)}{2a};$$

$$\therefore \text{ Force acting to right} = F_1 - F_2 = \frac{W}{2a}\left[a(\mu_1 - \mu_2) - x(\mu_1 + \mu_2)\right];$$

$$\therefore \frac{W}{g}\frac{d^2x}{dt^2} = -\frac{W}{2a}(\mu_1 + \mu_2)x + \frac{Wa}{2a}(\mu_1 - \mu_2);$$

or

$$\frac{d^2x}{dt^2} = -\frac{g}{2a}(\mu_1 + \mu_2)x + \frac{g \cdot a}{2a}(\mu_1 - \mu_2).$$

Since the differential of a constant is zero, we may re-write the above equation as

$$\frac{d^2}{dt^2}(x-b) = -\frac{g}{2a}(\mu_1+\mu_2)[x-b],$$

where
$$b = \frac{\frac{g}{2a}\cdot a(\mu_1-\mu_2)}{\frac{g}{2a}(\mu_1+\mu_2)} = \frac{a(\mu_1-\mu_2)}{\mu_1+\mu_2}.$$

This represents a S.H.M. of period $2\pi\sqrt{\frac{2a}{g(\mu_1+\mu_2)}}$ about a centre distant from the axis of symmetry by the amount b, being to the right if b is +ve, i.e. $\mu_1 > \mu_2$, and to the left if b is −ve.

Alternatively:

Let $\mu_1 > \mu_2$. Then in the symmetrical position $R_1 = R_2$ and $F_1 > F_2$. The force acting to the right is $(F_1 - F_2)$.

If the beam is displaced by an amount y to the left,

R_1 increases, R_2 diminishes;

$\therefore$ $(F_1 - F_2)$ increases.

The rest position will be where $F_1 = F_2$,

i.e.
$$\mu_1 R_1 = \mu_2 R_2.$$

Let this distance be b from axis of symmetry.

$$R_1 = W\frac{(a-b)}{2a}, \qquad R_2 = W\frac{(a+b)}{2a};$$

$$\therefore \text{ for } F_1 = F_2, \quad \mu_1(a-b) = \mu_2(a+b),$$

or
$$b = \frac{(\mu_1-\mu_2)a}{\mu_1+\mu_2}.$$

If now the beam is displaced from this rest position by an amount x to the right,

R_2 is increased by $\frac{Wx}{2a}$,

R_1 is decreased by $\frac{Wx}{2a}$.

$$\left.\begin{aligned} \therefore\ F_2 &\text{ is increased by } \mu_2\cdot\frac{Wx}{2a} \\ F_1 &\text{ is decreased by } \mu_2\cdot\frac{Wx}{2a} \end{aligned}\right\} \text{ from their equal values;}$$

$\therefore$ Force acting is a restoring force (to the left) $= \dfrac{W}{2a}(\mu_1 + \mu_2) \cdot x$, proportional to the displacement.

$$\therefore \text{ S.H.M. of Period} = 2\pi\sqrt{\frac{2a}{g(\mu_1 + \mu_2)}}\,.$$

14. Two equal masses, of weight 1,000 lb. and radius of gyration 15 in., are keyed to opposite ends of a steel shaft, 24 in. long. The shaft is 3 in. diameter for the first 10 in. of its length ; 5 in. diameter for the next 4 in. of its length ; and 3·5 in. diameter for the remainder of its length. Determine the frequency of the free torsional vibrations of the system and the position of the node. [Liv. U.]

Take $C = 12 \times 10^6$ lb. per sq. in.

Ans. 22·75 per sec. Node distant 7·98 in. from the free end of the 3 in. diam. section.

15. The sketch represents a system of two loads, W_1 and W_2, which can perform longitudinal oscillations in the vertical plane. Fig. (*f*).

Determine the two natural frequencies possible, given :

$W_1 = 320$ lb. $F_1 = 1{,}000$ lb. per ft.
$W_2 = 160$ lb. $F_2 = 5{,}000$ lb. per ft.
$F_3 = 1{,}250$ lb. per ft.

[Liv. U.]

Ans. 2·94 and 1·22 per sec.

16. In longitudinal vibration :

If f_1 is the frequency of a load W_1 on a spring of stiffness F_1 ;
f_2 is the frequency of a load W_1 on a spring of stiffness F_2 ;
f_3 is the frequency of a load W_2 on a spring of stiffness F_2 ;
and f_4 is the frequency of a load W_2 on a spring of stiffness F_3 ;

show that the frequencies (f) of the two-load system shown in Fig. (g) are given by

$$(f^2)^2 - f^2[f_1^2 + f_2^2 + f_3^2 + f_4^2] + [f_1^2 + f_2^2][f_3^2 + f_4^2] - f_2^2 f_3^2 = 0.$$

[Liv. U.]

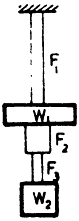

FIG. (*f*).

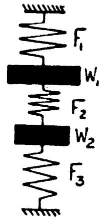

FIG. (*g*).

An overhung shaft consists of two sections, section (1) starting from the bearing. At the end of this section is situated a flywheel of moment of inertia I_1, and a second flywheel I_2 is situated at the free end of the second section.

If the frequencies of natural torsional oscillation of I_1 only on shaft section (1) only, I_2 only on section (1) only, and I_2 only on section (2) only, are 1, 2 and 3 vibrations per sec., determine the frequencies of the two natural modes of the whole system.

Ans. 13·325 or 0·676 per sec.

17. Determine the two possible frequencies of longitudinal oscillation of the system shown in Fig. (*h*), given :

$W_1 = 384$ lb. $F_1 = 300$ lb. per ft.
$W_2 = 192$ lb. $F_2 = 300$,, ,,
$F_3 = 200$,, ,,
$F_4 = 240$,, ,, [Liv. U.]

Ans. 1·15 and 1·63 per sec.

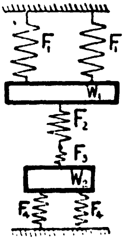

FIG. (*h*).

18. A uniform shaft, 2 in. in diameter, carries a series of flywheel loads as shown in Fig. (*i*). Determine approximately (*a*) the position of the node for the fundamental frequency of torsional oscillation, and (*b*) the fundamental frequency.

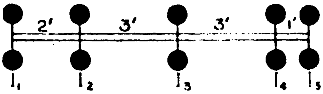

FIG. (*i*).

$I_1 = 4{,}000$ lb. ft.^{2} $I_4 = 2{,}000$ lb. ft.^{2} $C = 12 \times 10^6$ lb. per sq. in.
$I_2 = 6{,}000$,, $I_5 = 3{,}000$,,
$I_3 = 1{,}000$,, [Liv. U.]

Ans. 2·04 sec.

19. A vertical rod, 2 in. diameter and fixed at one end, carries three masses of 500, 300, and 200 lb., situated at distances of 1 ft., 3 ft., and 5 ft. from the fixing. Calculate approximately the fundamental frequency of longitudinal vibration. Take $E = 30 \times 10^6$ lb. per sq. in. [Liv. U.]

Ans. 179 per sec.

20. The circular frequency of a single load W supported by a spring of stiffness F, which is fixed at its upper end, is 10 radians per sec. Determine approximately the fundamental frequency of four similar systems which are attached one below the other in series. *Ans.* $\sqrt{10}/2\pi$.

✓**21.** Fig. (*j*) represents a load W attached by a spring of stiffness F_1 to a belt, which passes over a disk, of moment of inertia I and radius R, and is finally anchored by the spring of stiffness F_2. Assuming that no slip occurs in the belt for small angular displacements of the disk, determine the frequencies of the two natural modes when

$W = 100$ lb.
$I = 225$ lb. ft.^{2} $R = 1{\cdot}5$ ft.
$F_1 = 1{,}000$ lb. per ft.
$F_2 = 1{,}500$ „ „

Ans. 4·94 or 2·01 cy. p.s.

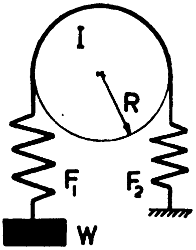

Fig. (*j*).

This problem, and the next, may be solved direct, or, if preferred, the disks can be first converted into equivalent masses in linear oscillation, i.e. $m = I/R^2$; show that this is so.

✓**22.** Fig. (*k*) represents two disks, of moments of inertia I_1, I_2 and radii R_1, R_2 respectively, which are free to turn about their centres O_1, O_2. Their motions are controlled by belts which are coupled to springs of stiffnesses F_1, F_2, F_3 as shown, and the initial tension throughout the system is such that for any small angular movements the springs will remain in tension and there will be no slip. Determine the frequencies of the two natural modes of vibration if

$I_1 = 3{,}200$ lb. ft.^{2} $F_1 = 1{,}000$ lb. per ft.
$I_2 = 6{,}400$ „ „ $F_2 = 2{,}000$ „ „
$R_1 = 1$ ft. $F_3 = 4{,}000$ „ „
$R_2 = 2$ ft. [Liv. U.]

Ans. 1·80 or 0·746 per sec.

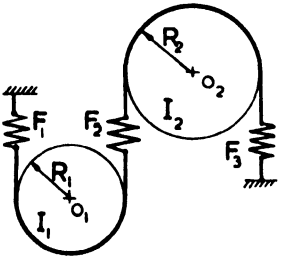

Fig. (*k*).

23. Explain the meaning of (*a*) degree of freedom, (*b*) natural mode, and (*c*) normal elastic curve in regard to a system in vibration.

A shaft system, free to revolve, carries a series of four concentrated loads. The first load has a moment of inertia of 16,000 lb. ft.^{2} units and is coupled to the next by a solid circular shaft of length 3 ft. and diameter 2 in. [$C = 12 \times 10^6$ lb. per sq. in.]

The ratio of the amplitudes of these two loads for the three modes of torsional oscillation possible is

$$\frac{\theta_2}{\theta_1} = +0{\cdot}8, \quad +0{\cdot}4, \quad -0{\cdot}5.$$

Determine the corresponding frequencies. [Liv. U.]

Ans. 0·67, 1·16, 1·83 per sec.

24. Given that 13·80 vibrations per second is the frequency of one of the natural modes of torsional oscillation of the shaft system sketched [Fig. (*l*)], determine the number and position of the nodes. The inertia of the shaft may be neglected.

$I_1 = 3{,}000 \dfrac{\text{lb.}}{g}\text{ in.}^2$
$I_2 = 5{,}000$,,
$I_3 = 1{,}600$,,
$I_4 = 2{,}500$,,

The torque per radian of twist of each shaft section is
(1) $7{\cdot}186 \times 10^6$ lb. in.
(2) $6{\cdot}288 \times 10^6$,,
(3) $3{\cdot}182 \times 10^6$,,

[Liv. U.]

Ans. 3 nodes : Shaft (1) 13·5 in. from load (1)
,, (2) 11 in. ,, ,, (2)
,, (3) 25¼ in. ,, ,, (3)

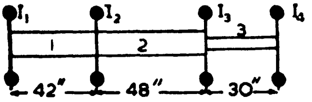

FIG. (*l*).

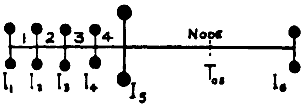

FIG. (*m*).

25. A 6-cylinder aero-engine is direct coupled to an airscrew. The moment of inertia of each equivalent cylinder disk is $0{\cdot}15 \dfrac{\text{lb. in.}^2}{g}$ and that of the airscrew $25 \dfrac{\text{lb. in.}^2}{g}$ units. The stiffness of each shaft section between consecutive cranks is $2{\cdot}8 \times 10^6$ lb. in. per radian and the stiffness of the connexion to the airscrew is $2{\cdot}02 \times 10^6$ lb in. per radian. Show that $1{,}000/2\pi$ is the 1-node frequency of torsional oscillation.

26. The following data refer to a four-cylinder marine diesel engine with flywheel I_5 direct coupled to a propeller I_6, Fig. (*m*).
Stiffness of shaft sections :

$$T_{01} = T_{02} = T_{03} = T_{04} = 50 \times 10^6 \text{ lb. ft. per radian ;}$$
$$T_{05} = 1 \times 10^6 \text{ lb. ft. per radian.}$$

Moment of inertia of propeller $I_6 = 64{,}000$ lb. ft.^{2} units.
Engine cylinder :

Mass of piston and crosshead = 5,440 lb.,
Mass of connecting rod = 2,400 lb.,

which may be divided equally between the reciprocating and the revolving masses at crosshead and crankpins respectively.* Moment of inertia of crank is equivalent to a mass of 600 lb. at crank radius of 1·25 ft.

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* See p. 214.

It is required that, for the fundamental one-node mode of torsional oscillation, the node should be situated midway between the flywheel and propeller.

Determine :

(*a*) the equivalent mass which, concentrated at the crankpin, will replace the reciprocating mass, and hence the moment of inertia of the whole rotating system per cylinder ;

(*b*) the one-node frequency ;

(*c*) the moment of inertia of the flywheel to give these conditions (use a tabular method of solution). [Liv. U.]

Ans. (*a*) $250 \frac{\text{lb.}}{g}$ ft.^{2}, (*b*) $\sqrt{1{,}000}/2\pi$, (*c*) $960 \frac{\text{lb.}}{g}$ ft.^{2}

✓**27.** Fig. (*n*) represents a system of three loads which can perform longitudinal vibrations in the vertical plane. Given the following data, what would be the stiffness of spring (3) in order that one of the frequencies should be $20/2\pi$ per second, and how many nodes are formed?

Spring stiffness :

$W_1 = 64$ lb. $F_1 = 400$ lb. per ft.
$W_2 = 96$ lb. $F_2 = 100$ lb. per ft.
$W_3 = 32$ lb.

[Liv. U.]

Ans. 267 lb. per ft.

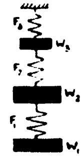


FIG. (*n*).

28. The sketch, Fig. (*o*), represents two shafts, of diameters 4 in. and 2 in., fixed at each end and carrying two pulleys, which are coupled by a non-elastic belt, in the positions shown. Determine the

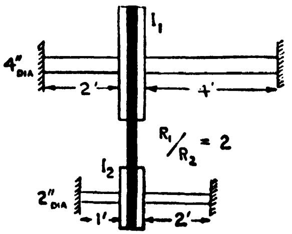


FIG. (*o*).

frequency of the natural torsional oscillations of this system if the moments of inertia of the pulleys are $I_1 = 32$ lb. ft.^{2}, $I_2 = 16$ lb. ft.^{2} units.

Take C for both shafts $= 12 \times 10^6$ lb. per sq. in. [Liv. U.]

Ans. 141 per sec.

29. Determine the natural frequencies of torsional oscillation of the geared system shown in Fig. (*p*), given :

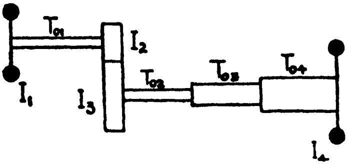


FIG. (*p*).

	Stiffness
$I_1 = 1{,}280$ lb. ft.^{2} units	$T_{01} = 25{,}000$ lb. ft. per radian
$I_2 = 2{,}000$,, ,,	$T_{02} = 200{,}000$,, ,, ,,
$I_3 = 2{,}240$,, ,,	$T_{03} = 300{,}000$,, ,, ,,
$I_4 = 5{,}120$,, ,,	$T_{04} = 600{,}000$,, ,, ,,

Gear ratio 2 : 1.

[Liv. U.]

Ans. 3·98 and 5·65 per sec.

30. Determine the frequencies of torsional oscillation of the geared system, Fig. (*q*), given that

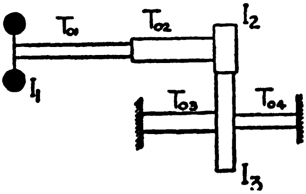


FIG. (*q*).

$I_1 = 6{,}400$ lb. ft.^{2}
$I_2 = 2{,}900$ lb. ft.^{2}
$I_3 = 2{,}700$ lb. ft.^{2}

Torsional stiffness of shaft sections :

$T_{01} = 10{,}000$ lb. ft. per radian
$T_{02} = 40{,}000$,, ,,
$T_{03} = 30{,}000$,, ,,
$T_{04} = 60{,}000$,, ,,

Gear ratio 3 : 1.

[Liv. U.]

Ans. 0·71 or 2·25 per sec.

31. Determine the two frequencies of natural torsional oscillation of the geared system shown in Fig. (*r*), given :

$I_1 = 64$ lb. ft.^{2} units.
$I_2 = 256$,, ,,
Inertia of gearing negligible.

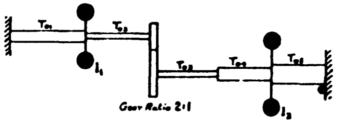


FIG. (*r*).

Torsional stiffness :

$T_{01} = 1{,}000$ lb. ft. per radian.
$T_{02} = 1{,}200$,, ,,
$T_{03} = 2{,}000$,, ,,
$T_{04} = 3{,}000$,, ,,
$T_{05} = 4{,}000$,, ,,
Gear ratio 2 : 1.

Reducing to speed of high speed shafting :

Equivalent stiffness of the three sections (2), (3) and (4) together

$$\frac{1}{T_0} = \frac{1}{1{,}200} + \frac{4}{2{,}000} + \frac{4}{3{,}000} \quad \text{or} \quad T_0 = 240 \text{ lb. ft. per rad.}$$

$$\text{Equivalent stiffness section (5)} = \frac{4{,}000}{4} = 1{,}000 \text{ lb. ft. per rad.}$$

$$\text{Equivalent inertia load (2)} = \frac{256}{4} = 64 \text{ lb. ft.}^2$$

The system thus becomes symmetrical, two loads of 64 lb. ft.^{2} on the three shaft sections of 1,000, 240, and 1,000 lb. ft. stiffness.

Solving by equations of motion gives

$$(n^2)^2 - 1240n^2 + 370{,}000 = 0,$$

or

$$n^2 = 740 \text{ or } 500.$$

Corresponding frequencies 4·33 and 3·57 per sec.

Solving by " nodes ", *only one value of x is obtained* which gives the true node obviously at the mid-point of the centre section. *How is the other frequency obtained?*

It will be that of the two loads moving in step as if the centre shaft were removed, i.e. a load (2 × 64) at junction of two shafts of stiffness 1,000 lb. ft. each.

32. Determine the natural frequencies of torsional oscillation of the system shown in Fig. (*s*), neglecting the inertia of the gearing.

Given :

Moment of inertia,

$$I_a = I_b = \tfrac{1}{10} I_c = 160 \text{ lb. ft.}^2 \text{ units.}$$

Stiffness,

$$T_{0x} = T_{0y} = 5{,}000 \text{ lb. ft. per radian.}$$

$$T_{0z} = 6{,}400 \text{ lb. ft. per radian.}$$

Speed of shaft z is one-quarter that of x or y.

Ans. 2·02 and 5·03 per sec.

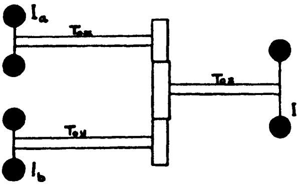


FIG. (*s*).

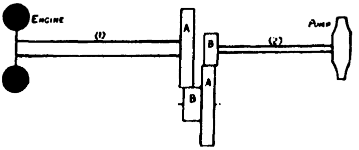


FIG. (*t*).

As the two branches I_a on shaft (x) and I_b on (y) are exactly similar, instead of leaving them as branches in our equivalent system, we can join them together so forming one load $= 2I_a$ on a shaft of stiffness $2T_{0x}$, so that the system reduces to that of two loads of 320 lb. ft.^{2} and $1{,}600 \times (\frac{1}{4})^2$ lb. ft.^{2} at the ends of a single shaft, the equivalent stiffness of which will be found to be $\frac{10{,}000}{26}$ lb. ft. per rad.

But this means that there is only one natural frequency, whilst if the branches had been kept separate there would have been two frequencies. *What is the missing one? It should be clear that it will be I_a oscillating against I_b, a node being formed at the gear box*; in other words it is that of I_a on the shaft (x).

If the inertia of the gearing had to be taken into account, it is much simpler to proceed by the above method, the system reducing to that of three loads giving a quadratic for the frequency equation, and find the third frequency as explained. By keeping the branch in, a cubic will result giving of course all three frequencies.

33. An engine drives a centrifugal pump through the double gearing sketched in Fig. (*t*). Determine the natural frequencies of the system, given :

Moment of inertia of engine equivalent rotating system, 16,000 lb. ft.^{2}

Moment of inertia of pump runner, 400 lb. ft.^{2}

Moment of inertia of each gear-wheel (A), 320 lb. ft.^{2}

Moment of inertia of each pinion (B), 80 lb. ft.^{2}

Speed of each pinion is twice that of its gear-wheel.
Torsional stiffness of engine shaft (1), 20,000 lb. ft. per radian.
Torsional stiffness of pump shaft (2), 5,000 lb. ft. per radian.

[Liv. U.]

Ans. 1·55 and 5·85 per sec.

34. Determine the natural frequencies of torsional oscillation of the geared system shown in Fig. (*u*), in which the inertia of the gears can be neglected.

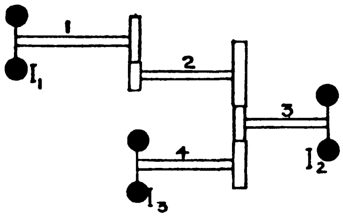


Fig. (*u*).

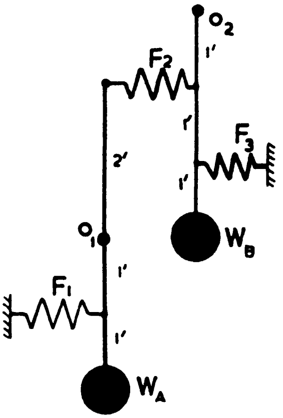


Fig. (*v*).

$I_1 = 57{,}600$ lb. ft.^{2}
$I_2 = 3{,}200$,,
$I_3 = 6{,}400$,,

$T_{01} = 36{,}000$ lb. ft. per radian
$T_{02} = 9{,}000$,, ,, ,,
$T_{03} = 4{,}000$,, ,, ,,
$T_{04} = 2{,}000$,, ,, ,,

Speed ratios $\frac{N_2}{N_1} = 2, \ \frac{N_3}{N_2} = 3, \ \frac{N_3}{N_4} = 2.$

[Liv. U.]

Ans. $4/2\pi$, $\sqrt{10}/2\pi$.

This arrangement will be found to reduce to a symmetrical branch system which can therefore be reduced still further.

35. Fig. (*v*) represents two pendulums pivoted at the points O_1, O_2 respectively, and controlled by the springs F_1, F_2, F_3.

Determine the two natural frequencies, given that

$W_A = 384$ lb.
$W_B = 21\frac{1}{3}$ lb.

$F_1 = 2{,}400$ lb. per ft.
$F_2 = 120$,, ,,
$F_3 = 120$,, ,,

Transfer the masses and springs into the line of the spring F^2. The equivalent system is then identical with that of Example 17.

CHAPTER IV

DAMPED VIBRATIONS

We have so far assumed that no frictional resisting forces act on the systems in free vibration, but actually, in practice, such forces, however small, are always present and will in time damp out the vibrations. The resistance offered by air or oil friction to the motion of a body is very complex, no simple law being known which will hold for all velocities. For moderate velocities (such as would be produced by a loaded spring vibrating in air or coupled to an oil dashpot) friction may be assumed to vary as the velocity, which lends itself to mathematical investigation, but for very high velocities friction appears to vary as the square of the velocity, in which case no mathematical solution can be obtained.

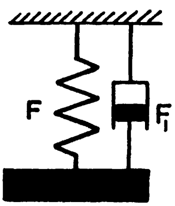


Fig. 40.

Assuming, then, that we have a single load supported by an elastic structure, which is immersed in a medium whose resistance varies as the velocity, the equation of motion will be

$$\underset{\substack{|\\\text{Force}\\\text{accelerating}\\\text{load}}}{\frac{W}{g}\frac{d^2x}{dt^2}} = \underset{\substack{|\\\text{Force exerted}\\\text{by elastic}\\\text{constraints}}}{-Fx} \underset{\substack{|\\\text{Resisting}\\\text{force of}\\\text{friction}}}{-F_1\cdot\frac{dx}{dt}},$$

or

$$\frac{W}{g}\frac{d^2x}{dt^2} + \underset{\substack{|\\\text{Force to}\\\textit{overcome}\\\text{friction}}}{F_1\frac{dx}{dt}} + \underset{\substack{|\\\text{Force to}\\\text{overcome}\\\text{elastic}\\\text{constraints}}}{Fx} = 0,$$

where F_1 is the resisting force per unit velocity or " damping coefficient ", the resistance, of course, always opposing the motion. Fig. 40 shows such an arrangement with a dashpot attachment.

Thus

$$\frac{d^2x}{dt^2}+2k\frac{dx}{dt}+n^2x=0, \quad\text{.......................}(1)$$

where

$$2k=\frac{F_1g}{W},$$

$$n^2=\frac{Fg}{W} \text{ as before.}$$

The modern method of solving differential equations is by means of the " operator " and will be familiar to some students, but for those unacquainted with the method the physical interpretation will probably be clearer if we use the older idea of making a standard substitution. Although not necessary for the form of solution which we propose to adopt, the constant $2k$ has been employed in order to conform with the operator method.

The standard substitution for such a differential equation would be $x=\epsilon^{\lambda t}$, where t represents time and λ is some constant, because upon differentiation a similar function results, e.g. $\frac{dx}{dt}=\lambda\,.\,\epsilon^{\lambda t}$, $\frac{d^2x}{dt^2}=\lambda^2e^{\lambda t}$, so that all the terms of the original equation can be expressed in terms of $e^{\lambda t}$ which considerably assists matters.

If, however, we make the substitution

$$x=e^{\lambda t}\,.\,y,$$

we shall obtain the solution in a form the meaning of which it is much easier to visualise.

We have then, differentiating the above,

$$\frac{dx}{dt}=\epsilon^{\lambda t}\left[\frac{dy}{dt}+\lambda y\right],$$

and

$$\frac{d^2x}{dt^2}=e^{\lambda t}\left[\frac{d^2y}{dt^2}+2\lambda\,.\,\frac{dy}{dt}+\lambda^2y\right].$$

Substituting in the original equation (1) we shall obtain, on collecting terms,

$$\frac{d^2y}{dt^2}+(2\lambda+2k)\,.\,\frac{dy}{dt}+[\lambda^2+2k\lambda+n^2]y=0.$$

Now, we have made no restrictions regarding the value of λ, hence let us make $\lambda = -k$, then the second term of the above equation will vanish and we have

$$\frac{d^2y}{dt^2} + [n^2 - k^2] \cdot y = 0.$$

Thus, $x = e^{-kt} \cdot y$ is the solution to the original equation (1), provided that the above equation in y is satisfied.

Three cases now occur :

(1) The most important case is where $n^2 > k^2$, so that the term $(n^2 - k^2)$ is real and may be denoted by μ^2. Thus

$$\frac{d^2y}{dt^2} + \mu^2 y = 0,$$

the solution of which we are already familiar with, being

$$y = a \cos (\mu t + \epsilon)$$

Hence the solution of the original equation is

$$x = e^{-kt} \cdot a \cos (\mu t + \epsilon).$$

The type of motion represented may be considered as a simple harmonic vibration, the amplitude of which diminishes exponentially to zero as t increases.

Let us examine the motion in detail, and in order to simplify matters slightly let us consider the phase angle ϵ to be zero, i.e. we start measuring time when $x = a$ instead of when $x = a \cos \epsilon$.

Thus $$x = e^{-kt} \cdot a \cos \mu t.$$

The space-time curve (Fig. 41) is included between the two curves $x = \pm ae^{-kt}$ since $\cos \mu t$ varies between ± 1, the points of contact A, B, C... occurring when

$$\mu t_1 = 0,$$

$$\mu t_2 = \pi, \text{ etc.,}$$

i.e. at equal intervals of time $t_2 - t_1 = \pi/\mu$.

Note that these are *not* the points of maximum displacement which occur *before* at M, N, etc. This will be explained below.

Now the value of x is zero every time $\cos \mu t$ is zero, i.e. when $\mu t_1 = \pi/2$, $\mu t_2 = 3\pi/2$, etc.

Thus the body arrives at its mid-position at equal intervals of time $(t_2 - t_1) = \pi/\mu$, or the time interval between two consecutive points on the mid-position line, when the body is moving in the *same* direction, such as D, E is $2\pi/\mu$.

Again, the velocity will be given by

$$\frac{dx}{dt} = ae^{-kt}(-\mu \sin \mu t - k \cos \mu t) \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

$$= \alpha e^{-kt} \cos(\mu t + \epsilon_1) \text{ as an alternative form,}$$

where

$$\left.\begin{aligned} \alpha \cos \epsilon_1 &= -ka \\ \alpha \sin \epsilon_1 &= \mu a \end{aligned}\right\} \text{ i.e. } \tan \epsilon_1 = -\frac{\mu}{k}.$$

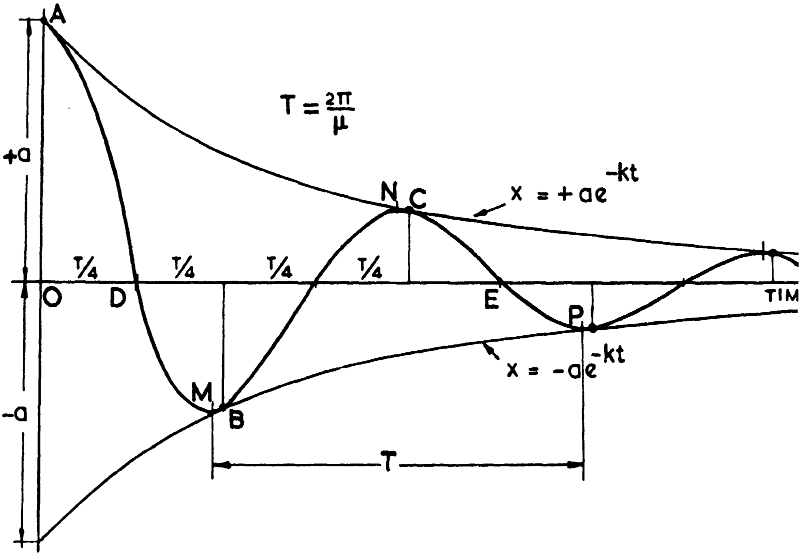


FIG. 41.

(These are merely the alternative forms of S.H.M. previously considered.) Thus the velocity-time curve is of similar form to that of the displacement, and since the body reaches the end of its swing on either side when $dx/dt = 0$, i.e. when $\cos(\mu t + \epsilon_1) = 0$, the time interval between two successive maxima of x on *the same side* of the mean, e.g. M, P, is $2\pi/\mu$.

Thus $2\pi/\mu$ is called the *Period*, and this interval can *only* be measured between consecutive crests or zero positions.

Now we have seen that S.H.M. can be represented by the projection of a vector which rotates with uniform angular velocity.

Here also we can consider a similar idea, the vector OP which generates the motion (Fig. 42) rotates with uniform angular velocity μ, but, instead of remaining constant in length, diminishes, according to the law

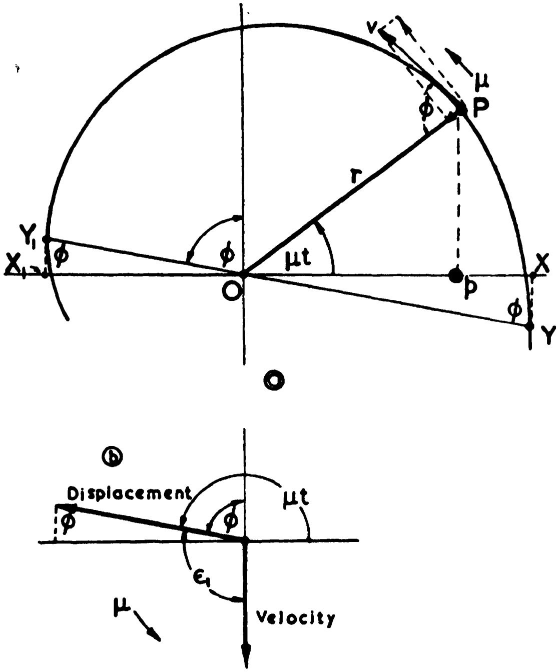


FIG. 42.

$$r = ae^{-kt},$$

and the displacement of the body from O is $x = r\cos\mu t$. The curve traced out is called an *equiangular spiral*, since the tangent to the curve always makes the same angle ϕ with the radius arm. (That this is so may be seen from the fact that the velocity of the generating point is made up of two components each proportional to the radius arm r—the one perpendicular to the arm $= r\mu$ due to the rotation of the arm, the other along the arm $= dr/dt = -kae^{-kt} = -k \, . \, r$. Hence their resultant will be proportional to r and will be inclined always at the same angle ϕ to the arm.) (Fig. 42*a*.)

Returning to equation (i) for the velocity, we noted that a maximum value of x occurs when $dx/dt=0$, i.e. when

$$\tan \mu t = -k/\mu.$$

This occurs when the tracing point on the displacement spiral is at either Y or Y_1 $(90° - \phi)$ *before* $\mu t=0$ or 180° and the projected points X, X_1, ... correspond with the points M, N, P... in Fig. 41.

Moreover, since the expression for the velocity is similar to that for the displacement but with phase difference ϵ_1, where $\tan \epsilon_1 = -\frac{\mu}{k}$, examination of the angles in Fig. 42*b*, in which the vectors are shown giving a maximum displacement and a corresponding zero velocity, shows that

$$\epsilon_1 = 180° - \phi,$$

so that $$\tan \epsilon_1 = -\tan \phi,$$

and $$\therefore \tan \phi = \frac{\mu}{k}.$$

Note that, with the same argument as for the displacement vector, the body will possess its maximum velocity *before* reaching its mean position.

Now, if at a time t_1 the body is at the end of its swing, its amplitude is given by

$$x_1 = ae^{-kt_1} \cos \mu t_1.$$

After an interval $t_2 - t_1 = \frac{\pi}{\mu} = \frac{1}{2}$ period the body will be at the other extreme end of its swing, and its amplitude is now

$$x_2 = ae^{-kt_2} \cos \mu t_2.$$

But for these conditions (i.e. points Y, Y_1, Fig. 42*a*),

$$\cos \mu t_1 = -\cos \mu t_2\,;$$

$$\therefore \frac{x_2}{x_1} = -e^{-k(t_2-t_1)} = -e^{-k\pi/\mu}.$$

This is termed the *Amplitude Reduction Ratio.*

Similarly, if x_3 is the next maximum displacement on the *same* side of the mean as x_1,

$$\frac{x_3}{x_1} = +e^{-2k\pi/\mu} \quad \text{since } t_3 - t_1 = \frac{2\pi}{\mu}.$$

This ratio is the same for all pairs of consecutive maxima on the same side of the mean and the quantity

$$\frac{2k\pi}{\mu}\left(=-\log_e\frac{x_3}{x_1}\right)$$

is called the *logarithmic decrement* of the motion, and is important since it enables us to determine the damping coefficient experimentally by noting this reduction ratio and the period.

Now for ordinary S.H.M. without damping, i.e.

$$\frac{d^2x}{dt^2}+n^2x=0,$$

the period is

$$\frac{2\pi}{\sqrt{n^2}}.$$

When damping is introduced we have seen that the period is increased to

$$\frac{2\pi}{\sqrt{\mu^2}}=\frac{2\pi}{\sqrt{n^2-k^2}}.$$

Thus if the frictional coefficient $2k$ is small compared with n, (e.g. systems vibrating in air), the two periods will only differ by a small amount of the second order, the chief effect being on the amplitude. The angle ϕ would then be very nearly 90°, so that the maximum displacement would occur for all practical purposes when $\cos\mu t=\pm 1$.

Case (2). $n^2=k^2$.

The equation

$$\frac{d^2y}{dt^2}+[n^2-k^2]y=0$$

becomes

$$\frac{d^2y}{dt^2}=0.$$

Hence

$$y=(At+B),$$

and

$$\therefore\ x=e^{-kt}.(At+B).$$

Now $B=$ the initial displacement when $t=0$, and A depends on the initial velocity which the body possesses when we start to measure time, e.g.

$$\frac{dx}{dt}=e^{-kt}[A-k(At+B)],$$

When $t=0$,

$$\frac{dx_0}{dt}=[A-kB],$$

or

$$A=\frac{dx_0}{dt}+kB.$$

If, when time is measured, the body is passing its zero position, i.e. $B=0$, then the initial velocity $=A$.

If the body is displaced a distance B and then released without being given any starting impulse the initial velocity will be zero and $A=kB$.

Now the starting impulse (if any) may be given in the direction of the displacement B or against, i.e. either away from or towards the mean position, and the sign of the initial velocity will depend on this. Hence A and B may have the same or opposite sign according to the magnitude and sign of dx_0/dt.

(*a*) If of the *same* sign, $x=0$ can only occur when $t=\infty$ since the diminution of the first factor e^{-kt} ultimately prevails over the increasing second factor $(At+B)$.

(*b*) If of *opposite* sign, $x=0$ can *also* occur when $t=-\frac{B}{A}$.

Hence the body can only pass once through its mean position if at all.

Friction is excessive, and the motion is not an oscillation, being termed *Aperiodic*.

The curves (Fig. 43) indicate the motions possible.

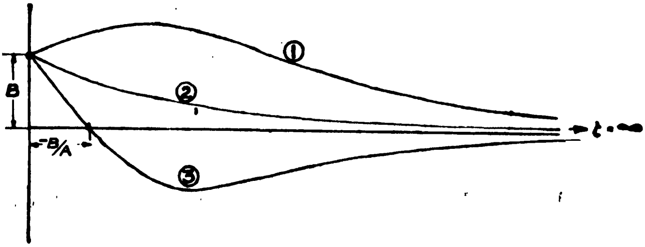


FIG. 43.

(1) Starting impulse away from mean position.
(2) No starting impulse.
(3) Starting impulse towards mean position.

Case (3). $n^2 < k^2$: The equation in y now becomes

$$\frac{d^2y}{dt^2} - \mu^2 y = 0,$$

where $$\mu^2 = k^2 - n^2.$$

The solution of such an equation is

$$y = Ae^{\mu t} + Be^{-\mu t},$$

whence $$x = Ae^{-\lambda_1 t} + Be^{-\lambda_2 t},$$

where A and B are constants depending on the initial conditions,

and $$\left.\begin{aligned} \lambda_1 &= k - \sqrt{k^2 - n^2} \\ \lambda_2 &= k + \sqrt{k^2 - n^2} \end{aligned}\right\} \text{ and are both positive.}$$

The value of x tends ultimately to zero with increase of time, and the motion is Aperiodic.

For the body to pass through its mean position, i.e. $x = 0$, we have

$$e^{(\lambda_2 - \lambda_1)t} = -\frac{B}{A}.$$

A real value of t can only occur if A and B are of opposite sign, which will depend on the initial displacement and velocity, so that the body can only pass once, if at all, through its mean position.

While Case (1) is the most important from our point of view as a vibration can take place, the transitional conditions, Case (2), are convenient as a basis of comparison for damping coefficients. Thus for these *critical damping* conditions let the damping coefficient be $F_{1c} = 2k_c \cdot \frac{W}{g}$, i.e.

$$k_c = n = \sqrt{\frac{Fg}{W}}$$

for a single load in longitudinal vibration. Then, for any other damping coefficient F_1

$$\frac{F_1}{F_{1c}} = \frac{2k}{2k_c} = \frac{k}{n}.$$

This ratio will appear in many expressions in the following chapters.

EXAMPLES II

CHAPTER IV

1. A load supported by an elastic structure performs a damped oscillation of period 1 second in a medium, the resistance of which varies as the velocity. At a given instant the amplitude was observed to be 10 cms. and in 10 secs. this had been reduced to $\frac{1}{10}$ cm.

What would be the period of the free vibrations if the resistance of the medium were negligible? [Liv. U.]

Ans. 0·997 sec. ($k = 0{\cdot}4605$).

2. A load when attached to the end of a vertical spring, which is fixed at the other end, causes a static deflexion of 2 ft. A dashpot is fitted to the load, which causes a resistance varying as the velocity, and is adjusted so that critical damping conditions exist.

At a given instant the displacement of the load from its rest position was 1 ft. and in $\frac{1}{2}$ sec. the load was observed to pass through this position.

Determine

(1) the initial velocity of the load;

(2) the time taken to reach the maximum displacement on the other side of the mean, and

(3) the magnitude of this displacement.

Here
$$x = e^{-kt}(At + B)$$
$$x = 0, \text{ when } t = -B/A \text{ or infinity};$$
$$\therefore \tfrac{1}{2} = -B/A, \text{ and as } B = 1 \text{ ft.}, \therefore A = -2.$$

The initial velocity
$$\frac{dx_0}{dt} = A - kB$$
$$= -2 - k,$$

and k is given equal to n, i.e. $\sqrt{\frac{g}{\delta_{st}}} = \sqrt{\frac{32}{2}} = 4$;

$$\therefore \text{ initial velocity} = -6 \text{ ft. sec.}^2,$$

the negative sign indicating that it is directed towards the mid-position.

The body reaches its maximum displacement when the velocity becomes zero, i.e. when
$$A - k(At + B) = 0;$$
$$\therefore t = 3/4 \text{ sec.},$$
the displacement then being
$$e^{-4\cdot 3/4}(-2\,.\,3/4 + 1) = -0{\cdot}025 \text{ ft.}$$

3. A uniform tube closed at its base is weighted so as to float upright in a liquid, being immersed to a depth of 1 ft. when at rest. It is observed that, when the tube is given a small vertical displacement and released, a damped oscillation takes place in which the ratio of two consecutive maximum displacements on the same side of the mean is 10 : 1.

Assuming that the resistance of the liquid varies as the velocity, determine

(*a*) the value of k, and

(*b*) the period of the damped vibration.

[Liv. U.]

Ans. (*a*) 1·94; (*b*) 1·184 secs.

4. A single load on elastic connexions having a natural period undamped of $2\pi/20$ sec., is subjected to a resistance which varies as the velocity. It is given an initial displacement of 1 ft. and released without any starting impulse.

Determine the displacement after $\frac{1}{20}$, $\frac{1}{10}$, $\frac{1}{5}$ and $\frac{1}{2}$ sec. when (*a*) $k=n$ and (*b*) $k=2n$.

		(*a*)	(*b*)
Ans.	$\frac{1}{20}$ sec.	0·7358 ft.	0·6299 ft.
	$\frac{1}{10}$	0·4059	0·3676
	$\frac{1}{5}$	0·0916	0·1266
	$\frac{1}{2}$	0·000499	0·00504

For (*b*) we have

$$x = Ae^{-\lambda_1 t} + Be^{-\lambda_2 t},$$

$k=2n=40$ $\qquad \lambda_1 = k - \sqrt{k^2-n^2} = 10{\cdot}72,$

$$\lambda_2 = k + \sqrt{k^2-n^2} = 149{\cdot}28,$$

$t=0$ $\qquad$ initial displacement $x_0 = A + B = 1.$

$$\frac{dx}{dt} = -\lambda_1 Ae^{-\lambda_1 t} - \lambda_2 Be^{-\lambda_2 t};$$

$\therefore \dfrac{dx_0}{dt} = -\lambda_1 A - \lambda_2 B$ and is given here as zero;

$$\therefore \frac{B}{A} = -\frac{10{\cdot}72}{149{\cdot}28} = -0{\cdot}07185.$$

Whence $\qquad A = 1{\cdot}077,$

$$B = -0{\cdot}077.$$

With the values given, the second exponential term is negligible except for very low values of t. *Notice that the load is brought more nearly to rest in a shorter time in the case for critical damping, and this is always true for any conditions.*

CHAPTER V

NATURAL VIBRATIONS OF HEAVY SHAFTS (UNIFORM SECTION)

ON page 51 we showed how the inertia of a simple elastic system could be taken into account, but otherwise in all our calculations we have neglected the inertia of the elastic connexions in comparison with that of the loads.

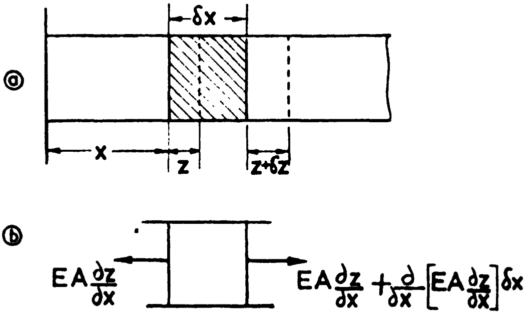

a
δx
x
z
$z+\delta z$
b
$EA\frac{\partial z}{\partial x}$
$EA\frac{\partial z}{\partial x}+\frac{\partial}{\partial x}\left[EA\frac{\partial z}{\partial x}\right]\delta x$

FIG. 44.

We will now consider the oscillations of a heavy shaft, unloaded except for its own mass, and will develop the equations for longitudinal oscillations, although in general the frequencies obtained in practice are so high that no trouble is likely to be experienced. Exactly similar equations can be deduced for the corresponding cases in torsional oscillation.

Consider then an element of the shaft enclosed between two sections distant x and $(x+\delta x)$ from one end (Fig. 44*a*).

At any instant let the longitudinal displacement during vibration of section x be z and that of $(x+\delta x)$ be $(z+\delta z)$, while the length δx of the element has become $\delta x+\frac{\partial z}{\partial x}\delta x$.

[For students who may be unfamiliar with " partial differentials " for which the symbol ∂ is used—the meaning is merely that $\frac{\partial z}{\partial x}$ is the rate of increase of z with x only, all other variables which may affect z being considered constant for this change.]

Now $\frac{\partial z}{\partial x}$ in words also means the alteration in length per unit length of the bar, i.e. measures the ***strain*** at the face x, so that the stress at this face will be $E \,.\, \frac{\partial z}{\partial x}$ or the force $= EA \,.\, \frac{\partial z}{\partial x}$ where A is the cross-sectional area, and E the modulus of elasticity.

If the shaft had no inertia this force would be constant across the element, but due to its inertia each element in vibration has to be accelerated so that there must be a difference in the forces on the two faces which were originally at distances x and $(x+\delta x)$.

This change of force we can express as a partial differential, e.g.

$$\frac{\partial}{\partial x}\left[EA\,\frac{\partial z}{\partial x}\right]\delta x = EA\,\frac{\partial^2 z}{\partial x^2} \,.\, \delta x,$$

where $\frac{\partial}{\partial x}\left[EA\,\frac{\partial z}{\partial x}\right]$ represents the rate of change of force with *distance* in the x direction only, or the change of force per unit distance, so that upon multiplying by δx we get the change over distance δx.

Thus the two faces are subjected to the forces shown in Fig. 44*b*.

Let $w =$ the *weight per unit length*, so that the mass per unit length in engineer's units is numerically equal to $\frac{w}{g}$.

Then the element δx has a mass of $\frac{w}{g}\,\delta x$ which is given an acceleration $\frac{\partial^2 z}{\partial t^2}$ (note that now we are considering change of z with *time* only) by the above difference in force, so that in the limit

$$\frac{w}{g}\,dx \,.\, \frac{\partial^2 z}{\partial t^2} = EA\,\frac{\partial^2 z}{\partial x^2} \,.\, dx,$$

or

$$\frac{\partial^2 z}{\partial t^2} = \frac{EAg}{w}\,\frac{\partial^2 z}{\partial x^2},$$

where z is a function of both x and t.

In all our previous work we have considered vibratory motion to be simple harmonic of the form $b\cos(nt+e)$ where the amplitude b is entirely independent of t.

Here then it would appear reasonable to assume that z can be written as $X\cos(nt+e)$, where X depends only on x (and not on t) and determines the configuration at the section x.

On this assumption we have

$$\frac{\partial^2 z}{\partial t^2} = -Xn^2\cos(nt+e)$$

[X being constant in this differentiation in regard to t.]

and $$\frac{\partial^2 z}{\partial x^2} = \cos(nt+e)\cdot\frac{d^2X}{dx^2};$$

[$\cos(nt+e)$ here constant when differentiating in regard to x, and $\frac{d^2X}{dx^2}$ is not "partial" as X depends only on x.]

$$\therefore \frac{EAg}{w}\frac{d^2X}{dx^2} + n^2X = 0,$$

or $$\frac{d^2X}{dx^2} + a^2X = 0,$$

where $a^2 = \frac{w}{EAg}\cdot n^2$.

This expression is of familiar form, except that for S.H.M. X would have been a function of time whilst here it is a function of distance x.

By analogy the solution will be

$$X = B\cos(ax+\epsilon)$$
$$= \alpha\cos ax + \beta\sin ax,$$

so that

$$z = [\alpha\cos ax + \beta\sin ax]\cos(nt+e),$$

where the constants α, β will be determined from the end conditions and type of problem.

Uniform cantilever. Suppose we wished to determine the frequencies of longitudinal vibration of a uniform cantilever of length l unloaded except for its own mass, we know that when $x=0$ at the fixing, the displacement z will be zero, and when $x=l$ at the free end there can be no stress in the rod, i.e. $\partial z/\partial x$ is zero, since the stress is proportional to this term as we have seen. Thus

$$\underline{x=0} \qquad\qquad z=0=\alpha\cos(nt+e).$$

If $\cos(nt+e)$ is zero, then the shaft would be at rest no vibration taking place, i.e. α must be zero ;

$$\therefore\ z=\beta\sin ax\,.\,\cos(nt+e).$$

$$\underline{x=l} \qquad\qquad \frac{\partial z}{\partial x}=0=\beta a\cos al\,.\,\cos(nt+e).$$

Now $\beta\neq 0$ as otherwise z would always be zero at every point, $\therefore$ $\cos al$ must be zero, so that

$$al=\frac{\pi}{2},\ \frac{3\pi}{2},\ \frac{5\pi}{2}\ \ldots,$$

i.e.
$$a=n\sqrt{\frac{w}{EAg}}=\frac{\pi/2}{l}\ \ldots.$$

There is thus an infinite number of frequencies possible, the fundamental being

$$\frac{n_1}{2\pi}=\frac{1}{2\pi}\,.\,\frac{\pi/2}{l}\sqrt{\frac{EAg}{w}}\,.$$

Since the Normal Elastic Curve gives the relative displacement or amplitude of the longitudinal vibration of any section, its equation will be $z_A=\beta\sin ax$, and substituting the above values of a, i.e. $\frac{\pi}{2l}$, $\frac{3\pi}{2l}$ ... we obtain the sine curves for the different modes shown in Fig. 45 where

(*a*) For the fundamental $z_A=\beta\sin\frac{\pi}{2l}x$.

When $x=l$ $\qquad z_A=\beta$.

No node (other than at fixed end).

(*b*) $z_A=\beta\sin\frac{3\pi}{2l}x$.

When $x=l/3$ $\qquad z_A=\beta$

$2l/3$ $\qquad z_A=0$

l $\qquad z_A=-\beta$

One node

and so on, one more node appearing for each higher frequency.

The student is left to work out the frequencies and normal elastic curves for a shaft fixed at both ends, and no difficulty should be experienced in converting the equations for the torsional oscillations of similar shaft arrangements.

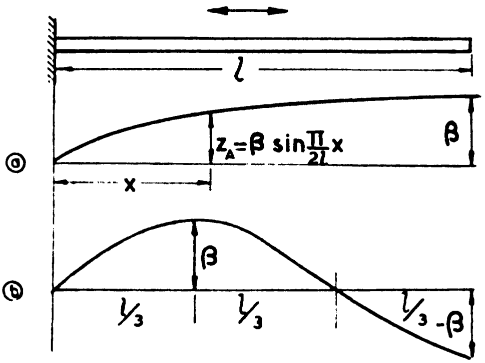


FIG. 45.

We shall replace

z by θ, the angular displacement at distance x ;

w by wk^2, the moment of inertia per unit length of shaft, k being the radius of gyration ;

A by J, the second moment of area of cross-section ;

and E by C, the modulus of rigidity.

so that

$$\left(\frac{w}{g} k^2 dx\right) \frac{\partial^2 \theta}{\partial t^2} = CJ \frac{\partial^2 \theta}{\partial x^2} \cdot dx \, ;$$

$$\therefore \frac{\partial^2 \theta}{\partial t^2} = \frac{CJg}{wk^2} \cdot \frac{\partial^2 \theta}{\partial x^2} ,$$

and $$\theta = [\alpha \cos ax + \beta \sin ax] \cos (nt + e).$$

Note also that $\frac{w}{A} \times 1 =$ weight per unit volume or density γ and $\frac{\gamma}{g}$ is usually denoted by ρ so that $a^2 = \frac{\rho}{E} n^2$, and it is interesting to note that $\sqrt{E/\rho}$ is the velocity of sound (or the propagation of waves) along the shaft (see page 268). Similarly, for torsion,

$$J=A\,.\,k^2\,;$$

$$\therefore\ \frac{CJg}{wk^2}=\frac{CAg}{w}=\frac{C}{\rho},$$

so that

$$a^2=\frac{\rho}{C}\,.\,n^2.$$

Longitudinal vibrations of a heavy cantilever carrying a concentrated load of weight W at its free end. At the free end of the shaft the force acting *on the shaft* will be

$$\left(EA\,\frac{\partial z}{\partial x}\right)_{x=l}$$

as deduced above. The stress at this element is now no longer zero, because of the attached load, which requires for its acceleration a force

$$\left(\frac{W}{g}\,\frac{\partial^2 z}{\partial t^2}\right)_{x=l},$$

so that the load will exert *on* the shaft an equal and opposite force, i.e.

$$\left(\frac{W}{g}\,\frac{\partial^2 z}{\partial t^2}\right)_{x=l}=-\left(EA\,\frac{\partial z}{\partial x}\right)_{x=l}\qquad\text{..................(i)}$$

We have already shown that, for a cantilever,

$$z=\beta\sin ax\cos(nt+e)\quad(\alpha\text{ being zero})$$

$$\therefore\ \frac{\partial^2 z}{\partial t^2}=-n^2\beta\sin ax\cos(nt+e),$$

and

$$\frac{\partial z}{\partial x}=\beta a\cos ax\cos(nt+e),$$

so that upon substituting in (i) and taking $x=l$,

$$-\frac{W}{g}\,n^2\sin al=-EA\,.\,a\cos al,$$

or

$$\tan al=\frac{EAga}{Wn^2}\,.$$

We can reduce this to another form as follows :

$$a^2=\frac{wn^2}{EAg},\quad\therefore\ \frac{EAg}{n^2}=\frac{w}{a^2};$$

$$\therefore\ \tan al=\frac{w}{Wa}.$$

If Y = the ratio of the weight of the shaft to the weight of the load, i.e. $=\frac{wl}{W}$ (where w = weight per unit length),

$$\therefore\ \underline{Y = al \tan al.}$$

The frequencies can be obtained from this equation graphically by taking various values of n^2 and plotting Y/al and $\tan al$ against al.

Example.

We have stated above that the frequencies of longitudinal vibration are so high that generally they are well beyond the operating range of any machine, so that no trouble from resonant effects can occur (see next chapter), but the analogous results in torsion are of great practical importance. Occasionally, however, longitudinal vibrations come under review and the following investigation was made by the author when trouble was experienced from axial vibrations of a marine propeller shaft system. The calculations are given in fair detail as they afford an excellent example on the genera' principles which we have been discussing.

A hollow marine-engine shaft carrying a 3-blade propeller at its free end was considered as a cantilever fixed at the thrust block as far as axial vibrations were concerned. To determine the effect on the frequency of " solidifying " the shaft over various lengths from the thrust block end (Fig. 46).

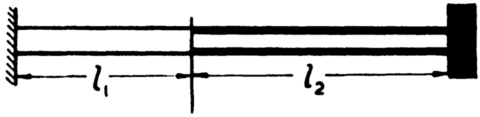


FIG. 46.

Let l_1 = solid length,

l_2 = hollow length,

then, with the same symbols as before, we shall have

$$\text{(i)}\quad \frac{\partial^2 z}{\partial t^2} = \frac{1}{a_1^2}\frac{\partial^2 z}{\partial x^2} \quad \text{for } x \geqslant 0, \text{ but } \leqslant l_1 \text{ (i.e. over length } l_1)$$

and
$$\text{(ii)}\quad \frac{\partial^2 z}{\partial t^2} = \frac{1}{a_2^2}\frac{\partial^2 z}{\partial x^2} \quad \text{for } x \geqslant l_1, \text{ but } \leqslant (l_1 + l_2) \text{ (i.e. over length } l_2).$$

Also from the previous sections we know that for a cantilever—

over length l_1: $z = \beta_1 \sin a_1 x \,.\, \cos(nt + e)$ $[\alpha_1 = 0]$

over length l_2: $z = [\alpha_2 \cos a_2 x + \beta_2 \sin a_2 x] \cos(nt + e)$.

At $x = l_1$:

$$\beta_1 \sin a_1 l_1 \,.\, \cos (nt+e) = (\alpha_2 \cos a_2 l_1 + \beta_2 \sin a_2 l_1) \cos (nt+e) \quad \ldots(1)$$

At $x = (l_1 + l_2)$:

$$\text{Force accelerating load} = \left(\frac{W}{g}\frac{\partial^2 z}{\partial t^2}\right)_{x=(l_1+l_2)} = -\left(E_2 A_2 \frac{\partial z}{\partial x}\right)_{x=(l_1+l_2)};$$

$$\therefore \quad -\frac{W}{g} n^2[\alpha_2 \cos a_2(l_1+l_2) + \beta_2 \sin a_2(l_1+l_2)] \cos (nt+e)$$
$$= -E_2A_2[-\alpha_2 a_2 \sin a_2(l_1+l_2)$$
$$+\beta_2 a_2 \cos a_2(l_1+l_2)] \cos (nt+e). \quad \ldots\ldots(2)$$

Also the force $EA\dfrac{\partial z}{\partial x}$ at the junction $(x = l_1)$ must be continuous :

$$\therefore \quad [E_1A_1\beta_1 a_1 \cos a_1 l_1] \cos (nt+e)$$
$$= [E_2A_2a_2(-\alpha_2 \sin a_2 l_1 + \beta_2 \cos a_2 l_1)] \cos (nt+e) \quad \ldots\ldots(3)$$

$\therefore$ $\left(\frac{3}{1}\right)$ gives $E_1A_1a_1 \cot a_1l_1$

$$= \frac{E_2A_2a_2(-\alpha_2 \sin a_2 l_1 + \beta_2 \cos a_2 l_1)}{\alpha_2 \cos a_2 l_1 + \beta_2 \sin a_2 l_1}, \quad \ldots\ldots\ldots\ldots\ldots(4)$$

thus eliminating β_1.

Then from (2) and (4) expressions for $\dfrac{\alpha_2}{\beta_2}$ are obtained which, upon equating, give

$$\frac{\alpha_2}{\beta_2} = \frac{Q \cos a_2(l_1+l_2) - \dfrac{W}{g} n^2 \sin a_2(l_1+l_2)}{Q \sin a_2(l_1+l_2) + \dfrac{W}{g} n^2 \cos a_2(l_1+l_2)}$$
$$= \frac{-P \cos a_1 l_1 \,.\, \sin a_2 l_1 + Q \cos a_2 l_2 \,.\, \sin a_1 l_1}{P \cos a_1 l_1 \,.\, \cos a_2 l_1 + Q \sin a_2 l_1 \,.\, \sin a_1 l_1},$$

where $P = E_1A_1a_1,$

$Q = E_2A_2a_2.$

Using the trigonometrical relationships for the sine and cosine of the difference of two angles, the above expression reduces to

$$Q[P \cos a_1 l_1 \,.\, \cos a_2 l_2 - Q \sin a_1 l_1 \,.\, \sin a_2 l_2]$$
$$= \frac{W}{g} n^2 [P \cos a_1 l_1 \sin a_2 l_2 - Q \sin a_1 l_1 \cos a_2 l_2].$$

Now, the shaft being of the same material throughout, $E_1 = E_2$;

$$\therefore \quad a_1 = a_2 = \sqrt{\frac{\rho}{E}} \cdot n,$$

and thus $$P/Q = A_1/A_2,$$
so that

$$\frac{W}{g} n^2/Q = \frac{[A_1/A_2 \cdot \cot a_1l_1 \cdot \cos a_2l_2 - \sin a_2l_2]}{[A_1/A_2 \cdot \cot a_1l_1 \cdot \sin a_2l_2 + \cos a_2l_2]}, \quad \dots\dots\dots(5)$$

and finally, since a_2^2 is also equal to $\frac{w_2n^2}{E_2A_2g}$,

$$\frac{W}{g} n^2/Q = \frac{W}{g} n^2 E_2A_2a_2 = \frac{W}{g} \left(\frac{a_2^2g}{w_2}\right) \cdot \frac{1}{a_2} = \frac{Wa_2}{w_2} = \frac{Wa_2l_2}{w_2l_2}$$

$$= \frac{a_2l_2}{Y}, \text{ where } Y = \frac{w_2l_2}{W} = \frac{\text{weight of shaft } l_2}{\text{weight of propeller } W}.$$

This equation has now to be solved by plotting the two sides of the equation against values of a_2l_2.

Details of system.

Total length of shaft L	2175 in.
Total weight all solid	234,440 lb.
Total weight all hollow	106,940 lb.
A_1 (solid)	330 sq in.
A_2 (hollow)	123 sq in.
A_1/A_2	2·68
Weight of propeller	39,240 lb.
Density of steel	0·283 lb. per cu. in.
E for steel	30×10^6 lb. per sq. in.

Three cases were calculated:

(1) $l_1 = \frac{1}{4}L = \frac{1}{3}l_2 = 544$ in.

$l_2 = 1{,}632$ in.

w_2l_2 (wt. hollow length) $= 80{,}200$ lb. $\therefore \; Y = 2{\cdot}045.$

(2) $l_1 = \frac{1}{2}L = l_2 = 1{,}088$ in.

$w_2l_2 = 53{,}470$ lb. $\therefore \; Y = 1{\cdot}36.$

(3) $l_1 = \frac{3}{4}L = 3l_2 = 1{,}632$ in.

$l_2 = 544$ in.

$w_2l_2 = 26{,}735$ lb. $\therefore \; Y = 0{\cdot}631.$

$$a_2l_2 = n\sqrt{\frac{\rho}{E}} \cdot l_2 = nl_2\sqrt{\frac{{\cdot}283}{30 \times 10^6 \times 32{\cdot}2 \times 12}}$$

$$= \frac{nl_2}{202{\cdot}2 \times 10^3}.$$

Note that, if the whole shaft was solid or hollow throughout, i.e. $l_2 = 0$ and $A_1 = A_2$ so that P can be substituted for Q on the left-hand side of equation (5), this expression reduces to that previously obtained on page 137, which checks the result.

The results obtained from the graphical solution were as follows :

Shaft condition	*Fundamental frequency* $n/2\pi$	*Critical Engine Speed*
All hollow	1044 ~ per *Minute*	348 r.p.m.
Case (1)	1185 „ „	395 „
„ (2)	1308 „ „	436 „
„ (3)	1344 „ „	448 „
All solid	1206 „ „	402 „

The propeller, having three blades, will tend to set up three disturbing impulses per revolution so that if the engine was to run at a speed of $\frac{1}{3}$ of the frequency for the particular case considered, unwelcome resonant vibrations may be set up, as will be explained in the next chapter. The curve (Fig. 47) shows these critical speeds, and knowing the range of speed for which the engine is designed it might be possible to avoid resonance by suitably " solidifying " the shaft.

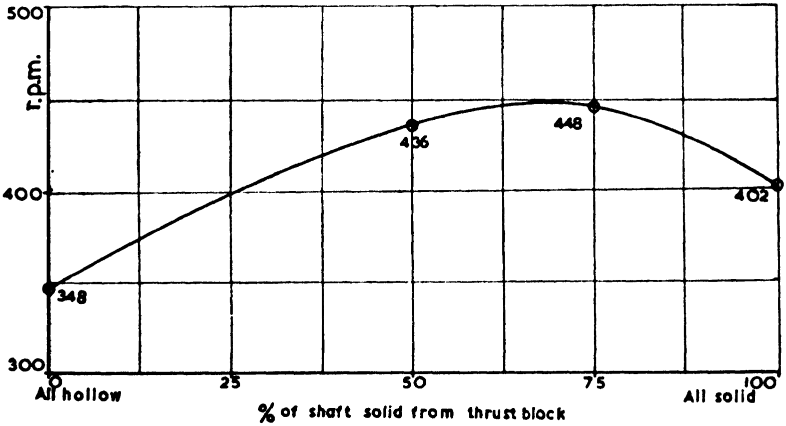


FIG. 47.

As an exercise the student might like to establish equation (5) working along the shaft in terms of " dynamic stiffness ", general expressions for the displacement z and the force at any section having already been deduced.

Extension of a heavy vertical rod due to its own weight. Consider a vertical rod fixed at its upper end. Using the same expressions as in the previous section, we have :

Strain at any section distant x from fixed end $=\frac{\partial z}{\partial x}$;

$\therefore$ Force at this section $=AE\,\frac{\partial z}{\partial x}=$ weight of shaft below this section $=w(l-x)$,

where $w=$ the weight per unit length,

$l=$ the length of the rod.

$$\therefore\ AEz=wlx-\frac{wx^2}{2}+c \qquad \left.\begin{aligned}\text{When } x&=0\\ z&=0\end{aligned}\right\}\ \therefore\ c=0$$

$$\therefore\ AE(z)_{x=l}=wl^2-\frac{wl^2}{2}=\frac{wl^2}{2};$$

$$\therefore\ \text{extension of rod } (z)_{x=l}=\frac{wl}{2}\cdot\frac{l}{AE},$$

i.e. is the same as would be produced by a concentrated load equal to half the actual weight of the rod if placed at the free end of a massless rod of the same elasticity.

CHAPTER VI

FORCED VIBRATIONS

We have now dealt with the free or natural vibrations that can be set up when an elastic system suffers a displacement due to an impulse or a sudden change in the magnitude of the load.

If this impact is not renewed the system will continue to vibrate in its own natural modes until the oscillations are damped out by frictional resistances, which, however small, are always present. If, however, the disturbing force is applied at regular intervals, i.e. if a *periodic force* acts on the system, the latter will be compelled to vibrate in a way foreign to it, depending on the nature of the disturbing force.

A disturbing force may be of two types: (*a*) impulsive, or (*b*) continuous, i.e. has a definite wave form.

If *impulsive*, such as might be produced by the blows from a hammer applied at regular intervals, then it can be visualised that between the applications of the impulses, the system will vibrate with its own free period, but the amplitude of the vibration will tend to increase and diminish with a rhythm depending on the relation between the frequency of the disturbing force and the natural frequency of the system, the impulses at one time coming into phase with the displacement, thereby increasing the amplitude, and then going out of phase and so opposing the motion.

If *continuous* say of *simple harmonic* form $P \cos (qt+e)$, and as will be shown this is the most important type for study, then we shall prove that the system will vibrate with S.H.M. of frequency $q/2\pi$ equal to that of the disturbing force, the magnitude of the amplitude depending on the relation between the forced and natural frequencies.

Critical frequency. If the frequency of the periodic force is *equal* to the natural frequency of the system upon which it acts, then it is clear that the force will always act in the direction of motion and never against, hence energy will continually be added to the vibrating system, causing the amplitude to increase until the elastic limit of the constraints is passed and fracture occurs. (If the force happens at the start to be applied so as to oppose the motion, it will merely bring the system to rest and start it off again with the forced frequency.) This value of the forced frequency equal to the natural is called the " *critical* ", " resonance " then occurring.

This is one of the vital problems which confronts the engineer. Every engineering structure is an elastic system which may be set into vibration by the application of a disturbing force which, if of the same frequency as the natural, may cause alarming amplitudes resulting in excessive stress and consequent breakdown.

It is for this reason that a regiment of soldiers is required to break step on crossing a suspension bridge, otherwise the rhythm of their marching might coincide with the natural frequency of the bridge.

The roll of a ship in the sea may attain alarming dimensions if acted upon by waves of the same frequency as the natural roll which would occur if the ship were given an angular displacement in calm water.

The balance weights in the driving wheels of a locomotive will produce a simple harmonic disturbing force in a vertical and horizontal plane. The vertical force might cause unwelcome vibrations in a bridge, which would also be subjected to a periodic disturbance due to the wheels passing over the joints in the track, etc.

The hull of a ship is an elastic structure which can vibrate in a variety of ways, and resonant conditions can easily be set up by unbalance in a reciprocating engine or due to a fluctuating turning moment on the crankshaft.

In order to illustrate the balancing of the reciprocating masses of a multi-cylinder engine, Professor Dalby designed a model 4-cylinder engine which was suspended from springs in a vertical plane, this being also the plane of reciprocation. The shaft was driven by an electric motor and any unbalance due to the reciprocating masses would produce a periodic disturbance in the plane of reciprocation, which would force a vibration of the engine on its springs.

By running the engine at such a speed that the frequency of the fluctuations of this unbalance coincided with the natural frequency of the engine on its springs, the amplitude of the vibration becomes very large, and by this means the slightest unbalance can be detected. At a higher speed, of course, the magnitude of the unbalanced force is greater, but the amplitude of the motion is less as we shall see. It is therefore only at this particular resonant speed that the engine should be run for the detection of any unbalance.

If the forced frequency is not exactly equal to the natural but in its neighbourhood, dangerous amplitudes may still be produced, since there may be enough applications of the disturbing force in the direction of motion to cause fracture, before becoming out of phase and opposing the motion.

Now we have stated that the nature or wave form of the disturbing force affects the type of vibration forced.

If " Simple Harmonic ", we have noted that a vibration having the same frequency as that of the disturbing force, and of constant amplitude, is set up, and resonance can only occur on coincidence of the forced and natural frequencies.

If Impulsive, however, it is easy to see that a disturbance of frequency 1/2, 1/3, ... , etc., of the natural frequency will cause resonance, since the force again is always applied in the direction of motion, the system having performed a complete number of vibrations and arrived into the same position by the time the next application of the force takes place.

Resonance, however, is not, in general, so dangerous with such a disturbing force, as friction, in between impulses, tends to damp down the vibration and prevent the amplitude becoming excessive.

Harmonic components. The wave forms of many disturbing forces that occur in practice—for instance, the turning moment of a reciprocating engine—are not simple harmonic in character, but any such curve can be split up into a Fourier series of sines and cosines, i.e. into harmonic components, the frequencies of which are usually 2, 3, 4, etc., times that of the actual disturbance.

[Fourier's Theorem states that any periodic function may be expanded as a series of the form

$$f(x) = A_m + A_1 \cos x + A_2 \cos 2x + \ldots + B_1 \sin x + B_2 \sin 2x \ldots,$$

or alternatively by

$$f(x) = A_m + a_1 \cos(x + \alpha_1) + a_2 \cos(2x + \alpha_2) + \dots$$

where $a_1 = \sqrt{A_1^2 + B_1^2}$ and $\tan \alpha_1 = -B_1/A_1$, etc.

A_m is the mean height of the curve, and the theorem means that the fluctuations about the mean can be split up into sine and cosine waves of frequencies 2, 3, 4, ... that of the fundamental.]

The resultant vibration set up is the vector sum of the vibrations which each of these components would cause if acting alone, and hence the effect of each harmonic can be considered separately.

It is thus clear that such a disturbing force, having a frequency of 1/2, 1/3, etc., of the natural frequency of the system upon which it acts, may cause dangerous amplitudes due to one of its harmonics synchronising with the natural frequency.

We shall deal with these harmonics in more detail in the following sections.

Note that though a disturbance of lower frequency may set up resonance in a system having a frequency which is an exact multiple of the other, the converse does *not* hold. For instance, if we had an elastic system with a natural frequency of say 10 vibrations per second acted upon by a disturbing force of frequency 20 vibrations per second, then if continuous, the harmonics produced will have frequencies of 40, 60, etc., vibrations per second, being still further removed from the natural ; while if the force was impulsive and the first application was in the direction of motion thus increasing the amplitude, the second application would meet the system on its return and so tend to stop the motion.

In certain cases the operating speed of a machine is designed to be well above the natural frequency, hence there will be no danger from harmonic components, and in running the machine to speed, the resonant conditions will exist for so short a time that no excessive stress will result.

The precise investigation of many actual problems that arise is difficult, but much may be learned from a study of the differential equations and their solutions representing the motion of a single load held in elastic constraints under the action of a simple harmonic disturbing force.

From experience the author has found that the vector method of solution—which will be familiar to those studying alternating current work—is by far the better method of illustrating the results, since it enables a visual picture to be obtained, which the mathematical process rather tends to conceal. (An excellent series of articles in support of the use of "Vector Methods of Studying Mechanical Vibrations" by Professor David Robertson, has appeared in *The Engineer*, Vol. CLI.)

First of all, however, we will study the mathematical solution.

Motion of a single concentrated load supported by an elastic structure and acted upon by a simple harmonic disturbing force.

Here, in addition to the restoring force of the constraints, we have acting a S.H. force of frequency (say) $\frac{q}{2\pi}$ and maximum value P, so that the force at any instant will be represented by $P\cos(qt+e)$.

The equation of motion is therefore

$$\underset{\substack{\big|\\ \text{Force}\\ \text{accelerating}\\ \text{load}}}{\frac{W}{g}\frac{d^2x}{dt^2}} = \underset{\substack{\big|\\ \text{Force}\\ \textit{exerted}\\ \text{by elastic}\\ \text{constraints}}}{-Fx} + \underset{\substack{\big|\\ \text{Disturbing}\\ \text{force}}}{P\cos(qt+e)}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

or

$$\frac{W}{g}\frac{d^2x}{dt^2} + \underset{\substack{\big|\\ \text{Force to}\\ \textit{overcome}\\ \text{elastic}\\ \text{constraints}}}{Fx} = P\cos(qt+e),$$

i.e.

$$\frac{d^2x}{dt^2} + n^2x = f\cos(qt+e), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

where

$$n^2,\ \text{as usual,} = \frac{Fg}{W},$$

$$f = \frac{Pg}{W}.$$

Now the only motion in which the acceleration and displacement have similar cosine forms is a S.H.M., so that it is reasonable to assume that the solution of the above equation, i.e. the resulting motion, is of the form

$$x = A \cos (qt + e).$$

Then $$dx/dt = -Aq \sin (qt + e),$$

$$d^2x/dt^2 = -Aq^2 \cos (qt + e).$$

Substituting in left-hand side of equation (2) we obtain

$$A\ (n^2 - q^2) \cos (qt + e).$$

Thus the solution holds provided

$$A\,(n^2 - q^2) = f,$$

or $$A = \frac{f}{n^2 - q^2},$$

i.e. $$x = \frac{f}{n^2 - q^2} \cdot \cos (qt + e).$$

Hence the given periodic force can maintain this forced oscillation of constant amplitude $\frac{f}{n^2 - q^2}$, depending on the forced and natural frequencies, and of frequency $\frac{q}{2\pi}$ equal to that of the disturbing force.

Now when the disturbing force is not acting, i.e. when the right-hand side of equation (2) is zero, we then have the simple harmonic solution of the form $a \cos (nt + \alpha)$, which represents the free oscillations that can exist independently of the forced, and when the disturbing force comes into play, the motion will be the resultant of these two, i.e.

$$x = \underbrace{a \cos (nt + \alpha)}_{\text{Free oscillations}} + \underbrace{\frac{f}{n^2 - q^2} \cos (qt + e)}_{\text{Forced oscillations}}.$$

The first term, i.e. the solution of the equation when the right-hand side $= 0$ is in mathematics termed the ***complementary function***, and the second term which concerns the forced motion only is called the ***particular integral***.

We have so far considered friction to be negligible, but such resisting forces, however small, are always present and the free oscillations will *in time* be damped out, leaving the system undergoing the forced oscillation only. At the start, however, these free oscillations may be important as, superimposed on the forced, a large amplitude, and therefore excessive stress may be produced.

The *total* displacement at any instant can be determined if, when we start to measure time, the initial displacement and velocity of the body are known, i.e. if a and α and A and e are known, or, in other words, if the lengths and initial positions of the two vectors, which generate the forced and free motions, are known. We have already examined such a motion on page 24, but it will be useful to consider the matter further.

Let us assume that when we start to measure time ($t=0$) the body is momentarily *at rest in its mid-position*, and for convenience let us take the phase angle $e=0$, which slightly simplifies the mathematics but in no way affects the motion, so that, using the alternative S.H. form for the free motion,

$$x = C\cos nt + D\sin nt + A\cos qt,$$

and when $t=0$,

$$x_0 = 0 \quad \text{and} \quad \frac{dx_0}{dt} = 0.$$

Thus

$$x_0 = 0 = C + A\ ; \quad \therefore\ C = -A = -\frac{f}{n^2 - q^2}\ ;$$

$$\frac{dx_0}{dt} = 0 = Dn\ ; \quad \therefore\ D = 0 \text{ as } n \neq 0.$$

Vectorially this means that (i) the *lengths* of the two generating vectors are C and A and are both equal, (ii) the vectors are initially at 180 degrees to one another and lie in the projection line, i.e. the body is at its maximum displacement in both the free and forced motion but on the opposite sides of the mean so that the resultant displacement, is zero, and (iii) being at the maximum displacements, the velocity in both motions will be zero. Fig. 48 illustrates these results, the projection line being vertical.

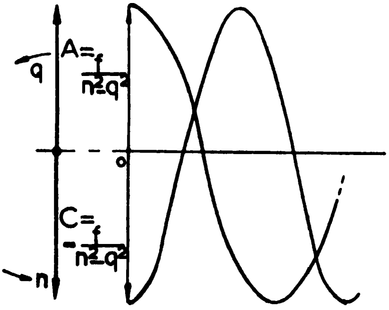


FIG. 48.

Hence at any instant

$$x = -\frac{f}{n^2 - q^2}\,[\cos nt - \cos qt]$$

$$= +\frac{2f}{n^2 - q^2}\sin\left(\frac{n+q}{2}\right)t \,.\, \sin\left(\frac{n-q}{2}\right)t.$$

We shall refer to this expression again shortly.

Assuming that the free oscillations have been damped out, consider the amplitude coefficient $\frac{f}{n^2 - q^2}$ of the forced motion.

When there is no *fluctuation* of the disturbance occurring, i.e. $q=0$, but a constant force P is applied

$$\text{Displacement } x = \frac{f}{n^2} = \frac{Pg}{W}\bigg/\frac{Fg}{W} = \frac{P}{F},$$

which, as we should expect, is the static displacement produced by this constant force P.

Now when the fluctuation of P takes place we have the forced motion

$$x = \frac{f}{n^2 - q^2}\cos(qt+e) = \frac{f}{n^2}\cdot\frac{1}{(1 - q^2/n^2)}\cdot\cos(qt+e)$$

$$= \frac{1}{(1 - q^2/n^2)}\cdot\frac{P}{F}\cdot\cos(qt+e)$$

$$= \frac{n^2}{n^2 - q^2}\cdot\frac{P}{F}\cdot\cos(qt+e).$$

The coefficient $\frac{1}{1 - q^2/n^2}$ or $\frac{n^2}{n^2 - q^2}$ is called the *magnification factor* or *dynamic magnifier*, since it shows how the actual amplitude differs from the displacement that would occur if the maximum value P of the disturbing force was applied *statically*.

Thus the "magnification factor" $= \dfrac{\text{Actual amplitude}}{\text{Amplitude under static force } P}$

i.e. $\frac{P}{F}$.

But $F \times$ actual amplitude $=$ Maximum force exerted in elastic structure,

so that the magnification factor $= \dfrac{\text{Maximum force exerted in structure}}{\text{Maximum value of disturbing force.}}$

The graph (Fig. 49) shows how this factor varies with the frequency ratio q/n.

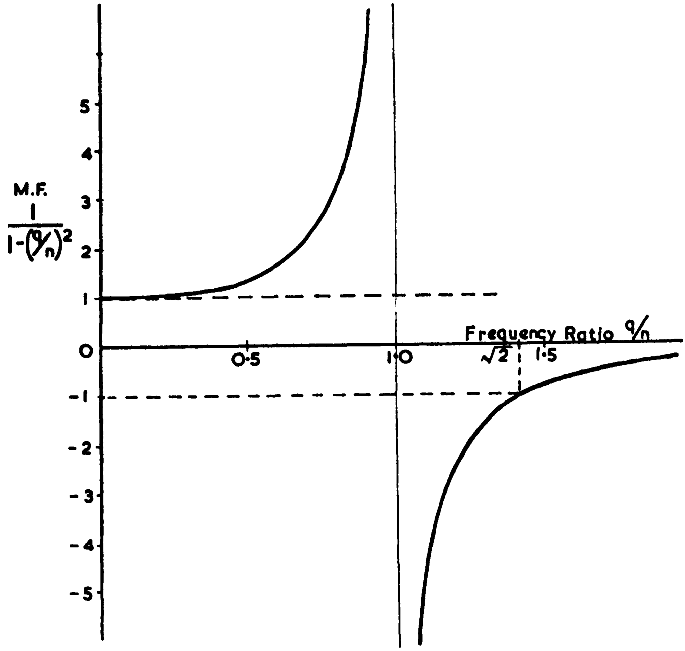


FIG. 49.

When q is very small compared with n, we see that the magnification factor approaches unity, i.e. the displacement of the load is approximately equal to the static extension of the structure under the disturbing force at any instant.

Again :

When q *is less than* n, the coefficient is positive, i.e. the displacement is in phase with the disturbing force.

As q *approaches* n the amplitude increases until when $q=n$ it would appear infinite, resonance then occurring.

When q *is greater than* n, the coefficient becomes negative, i.e. the displacement and force are of opposite phase, and as q continues to increase the amplitude becomes less and less until, for very high frequencies of the disturbing force compared with the natural, *the body may be assumed to be stationary in space.* This is an important result and will be considered in detail later.

Returning to our equation $x = \frac{f}{n^2 - q^2} \cdot \cos(qt + e)$, when $q = n$ this solution would appear to fail since it merely indicates that the amplitude is infinite, but gives no information as to the phase relationship of the displacement and disturbing force.

In order to simplify matters, we will assume the solution now holding and by substitution show that it is correct. The forced motion only is considered (particular integral).

Thus we have

$$\frac{d^2x}{dt^2} + n^2x = f \cos(qt + e).$$

When $q = n$:

Assume the solution $\underline{x = Bt \sin(qt + e)}$.

Then

$$dx/dt = Bt\,q \cos(qt + e) + B \sin(qt + e),$$

$$d^2x/dt^2 = -Bt\,q^2 \sin(qt + e) + 2Bq \cos(qt + e)\,;$$

$\therefore$ Substituting in L.H.S.,

$$\frac{d^2x}{dt^2} + n^2x = 2Bq \cos(qt + e),$$

remembering that $q = n$.

Hence $x = Bt \sin(qt + e)$ is the solution provided $2Bq = f$, i.e.

$$x = \frac{ft}{2q} \sin(qt + e),$$

which represents a vibration whose amplitude increases with the time, *the disturbing force being* 90° *in advance of the displacement* as is indicated by the cosine and sine terms, the force then always acting in the direction of motion causing the resonant conditions, and passing through its maximum as the load passes through its mean position.

Alternatively, we can establish the above result from the expression for the particular case of the general motion discussed on page 149.

We have

$$x = \frac{2f}{n^2 - q^2} \sin\left(\frac{n - q}{2}\right) t \,.\, \sin\left(\frac{n + q}{2}\right) t.$$

Let q very nearly equal n and let $n-q=2\Delta$ a very small quantity,

$$\therefore\ x=\left[\frac{f}{\Delta(n+q)}\,.\,\sin\Delta t\right]\sin\left(\frac{n+q}{2}\right)t.$$

Since Δ is small, $\sin\Delta t$ varies very slowly and its period $2\pi/\Delta$ is large. Thus the motion can now be considered as simple harmonic of frequency $\dfrac{n+q}{2}\Big/2\pi$ and amplitude

$$\frac{f}{\Delta(n+q)}\sin\Delta t$$

which varies very slowly, i.e. beats are occurring, the time interval between successive maxima or minima being $\frac{1}{2}$ period

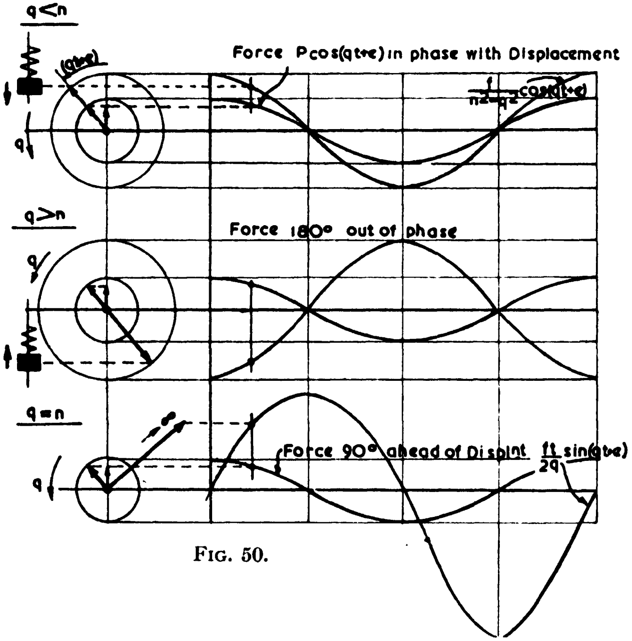


FIG. 50.

or $\dfrac{1}{2}\cdot\dfrac{2\pi}{\Delta}=\dfrac{2\pi}{n-q}$, a result with which we are already familiar.

When $q=n$, in the limit we can substitute Δt for $\sin\Delta t$ and thus avoid making the expression for the amplitude indeterminate $\left[=\dfrac{0}{0}\right]$.

Thus
$$x=\frac{ft}{2q}\sin qt.$$

(The phase angle e was assumed to be zero in establishing the above expression).

The wave forms representing the disturbing force and displacement fluctuations and the vectors generating them are shown in Fig. 50 for the different values of q assumed above.

Note that " force and displacement in phase or out of phase " does *not* mean that the force is acting in the same or opposite direction respectively as the motion of the load, but that the two generating vectors are together or opposed, the force, i.e. the projection of the force vector, sometimes being in the same direction and sometimes opposed to the motion, as inspection of the diagrams will show.

Note also that in taking the disturbing force as $P\cos qt$ when $t=0$ the force has its *maximum* value. If, when the force is applied, it rises from *zero* to its maximum value we must express the force as $P\sin qt$.

Forced vibrations—vector solution. In our differential equation above we have assumed that a simple harmonic disturbing force is acting, and then have shown that the motion produced is also simple harmonic of the same frequency.

We will now assume that the motion forced is simple harmonic and then deduce the nature of the disturbing force that must be applied to maintain this motion.

The general problem is therefore :

A body supported by an elastic structure, and having a natural frequency $n/2\pi$, is forced to vibrate with simple harmonic motion of a frequency $q/2\pi$, other than its natural, by the application of a disturbing force.

It is required to determine the nature of this disturbing force that must be applied to maintain the required motion.

Consider as an example the vertical vibrations of a loaded spring (Fig. 51*a*), and let the forced motion be

$$\underline{x=A\cos qt.}$$

As shown on page 17 this may be represented by the projection of the vector $oa'=A$ on the vertical line YY', rotation being with angular velocity q.

Similarly, the velocity

$$\frac{dx}{dt} = -Aq \sin qt$$

will be generated by the vector $od' = Aq$ situated 90 degrees in advance of oa', and the acceleration

$$\frac{d^2x}{dt^2} = -Aq^2 \cos qt$$

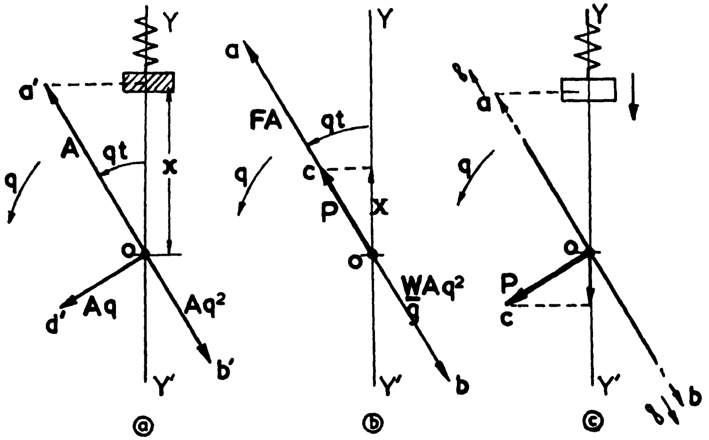


FIG. 51.

will be generated by the vector $ob' = Aq^2$, 180 degrees out of phase with the displacement vector oa'.

We will consider the two cases :

(*a*) Without damping.

(*b*) With damping varying as the velocity.

Thus the motion given is the solution of the equations respectively :

$$(a) \quad \frac{W}{g}\frac{d^2x}{dt^2} + Fx = X,$$

$$(b) \quad \underbrace{\frac{W}{g}\frac{d^2x}{dt^2}}_{\substack{\text{Force}\\ \text{accelerating}\\ \text{load}}} + \underbrace{F_1\frac{dx}{dt}}_{\substack{\text{Force to}\\ \textit{overcome}\\ \text{friction}}} + \underbrace{Fx}_{\substack{\text{Force to}\\ \textit{overcome}\\ \text{elastic}\\ \text{constraint}}} = \underbrace{X}_{\substack{\text{Disturb-}\\ \text{ing}\\ \text{force}}},$$

where X is the disturbing force whose nature has to be determined.

Now each of the terms on the left-hand side of either equation represents an *external force* acting on the body, and since each is proportional to either the displacement, velocity, or acceleration, of a motion that is given as simple harmonic, we may represent them at any instant by the projection of vectors parallel to the space vectors of Fig. 51*a*, which rotate with angular velocity q and have magnitudes equal to the maximum values of the respective forces. The sum of these vectors will give us the maximum value of X, and it is obvious that this resultant vector will rotate at the same angular velocity q, and its projection on the given diameter will give the value of the disturbing force at any instant. In other words, the disturbing force necessary to maintain the prescribed motion is also simple harmonic in character and of frequency $q/2\pi$.

Thus:

$$(a)\ \textit{No damping}: \quad \frac{W}{g}\frac{d^2x}{dt^2}+Fx=X.$$

Then, remembering we are considering *external forces* (Fig. 51*b*),

Vector $oa=FA=$ the maximum value of the force required to displace the body from its rest position through its amplitude A, i.e. the force required to *overcome* the elastic restoring force in this position.

This vector may also be taken to represent the position of the body at any instant, being proportional to the displacement vector oa'.

Vector $ob=\frac{W}{g}(Aq^2)=$ the maximum value of the force which gives the simple harmonic acceleration *towards* the mid-position.

This vector therefore acts in the opposite direction to oa, which represents a force acting away from the mid-position.

Thus the vector $oc=oa-ob=$ the maximum value P of the required disturbing force.

It is seen to be in phase with the displacement, i.e. coincides with the displacement vector oa' and is simple harmonic in character, so that

$$X=P\cos qt.$$

Hence the equation of which $x = A \cos qt$ is the solution is

$$\frac{W}{g}\frac{d^2x}{dt^2} + Fx = P \cos qt.$$

To determine the value of the amplitude A, we have

$$P = oa - ob = FA - \frac{W}{g} q^2 A\,;$$

$$\therefore\ A = \frac{P}{F - \dfrac{W}{g} q^2} = \frac{f}{n^2 - q^2} \text{ as before, where } f = Pg/W,$$

$$n^2 = Fg/W.$$

When q *is* $< n$, or $q^2 < \frac{Fg}{W}$, i.e. $\frac{W}{g} q^2 < F$ or $ob < oa$, the vectors are as shown in Fig. 51*b*, the disturbing force vector being in phase with the displacement vector *oa′*.

As q approaches n the amplitude A increases towards infinity.

For $q > n$, i.e. $ob > oa$, the disturbing force vector *oc* is now 180° out of phase with the displacement vector *oa′*.

For the transition stage when $q = n$ and A becomes infinite we argue as follows:

$P = oc$ is the maximum value of the disturbing force which is being applied and its value is therefore *fixed*, but, when $oa = ob$, P although still acting can now have no component along the line *ab*, and the only place for this vector, which will satisfy these conditions and cause the amplitude A to become infinite by continually adding energy to the system, is 90° ahead of *oa′*. In this position it will be seen (Fig. 51*c*) that the force at any instant (i.e. projection of *oc* or *YY′*) is always acting in the direction of motion of the load W.

Thus from this simple vector diagram we have been able to obtain the complete solution as before, and at the same time secure a visual interpretation of the results. We are, of course, considering the forced motion only (i.e. the Particular Integral), assuming that the free vibrations have been finally damped out.

(*b*) *With Damping varying as the velocity*:

$$\frac{W}{g}\frac{d^2x}{dt^2} + F_1\frac{dx}{dt} + Fx = X.$$

Again on the assumption that the motion is to be simple harmonic of amplitude A and frequency $q/2\pi$, we have in addition to the previous vectors *oa, ob,* a third vector *od* to represent the maximum force to be supplied to *overcome* friction (we are summing up the ***external*** forces that have to be supplied).

Its value is therefore F_1 (Aq) (the maximum value of the velocity being Aq), and its position will be in the direction of the velocity vector *od'*, i.e. 90° ***in advance of the displacement vector oa,*** since the force then always acts in the direction of motion, which is necessary as we have to overcome friction (Fig. 52*a*).

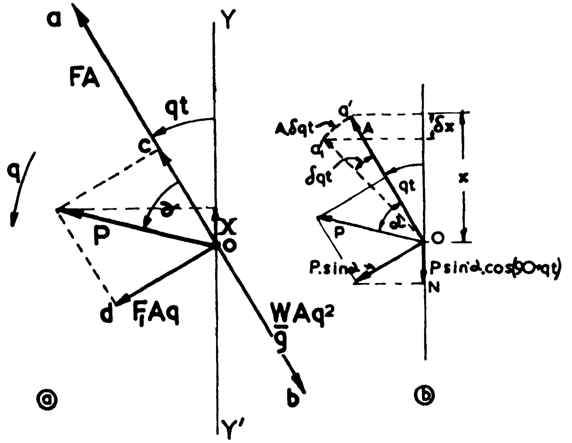


FIG. 52.

Hence the maximum value P of the disturbing force required is the resultant of *oc* and *od* ***and is at an angle α in advance of the displacement vector oa.***

If, then, the given motion is

$$x = A \cos qt,$$

the disturbing force required is

$$X = P \cos (qt + \alpha).$$

The angles in Fig. 52*a* show these conditions.

If, however, we make

$$X = P \cos qt,$$

the solution of the differential equation becomes

$$x = A \cos (qt - \alpha).$$

To determine the amplitude A and the phase angle α:

$$oc = P\cos\alpha = FA - \frac{W}{g}(Aq^2);$$

$$\therefore\ A = \frac{P\cos\alpha}{F - \dfrac{W}{g}q^2} = \frac{f}{n^2 - q^2}\cdot\cos\alpha, \quad \ldots\ldots\ldots\ldots\ldots(1)$$

and

$$\tan\alpha = \frac{od}{oc} = \frac{F_1(Aq)}{\left(F - \dfrac{W}{g}q^2\right)A} = \frac{F_1 q}{F - \dfrac{W}{g}q^2} = \frac{2kq}{n^2 - q^2}. \quad \ldots\ldots(2)$$

where, as before, $2k = \dfrac{F_1 g}{W}$.

Also from (1) and (2) we get

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{2kq}{n^2 - q^2};$$

$$\therefore\ \frac{\cos\alpha}{n^2 - q^2} = \frac{\sin\alpha}{2kq};$$

$$\therefore\ A = f\cdot\frac{\cos\alpha}{n^2 - q^2} = \frac{f\sin\alpha}{2kq}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Thus from (2) as q approaches n, α increases and when $q = n$, $\tan\alpha = \infty$, i.e.

$$\alpha = 90^\circ,$$

and the amplitude is $\dfrac{f}{2kq}$ from (3).

Hence on coincidence of the forced and natural periods, the disturbing force is 90° in advance of the displacement and is just sufficient to overcome friction at that speed and produce the above amplitude,

i.e. $$P = od = F_1 Aq.$$

Again, since $$\tan\alpha = \frac{2kq}{n^2 - q^2},$$

and $$\frac{1}{\cos^2\alpha} = 1 + \tan^2\alpha = \frac{(n^2 - q^2)^2 + 4k^2q^2}{(n^2 - q^2)^2};$$

$$\therefore\ A = \frac{f}{n^2 - q^2}\cdot\cos\alpha = \frac{f}{\sqrt{(n^2 - q^2)^2 + 4k^2q^2}}.$$

This expression could also have been derived from $P^2 = oc^2 + od^2$.

Now the static displacement of the system under the force P being

$$\frac{P}{F} = \frac{Pg/W}{Fg/W} = \frac{f}{n^2},$$

the expression for the amplitude may be written in the following form :

$$A = \frac{f/n^2}{\sqrt{\left(1 - \frac{q^2}{n^2}\right)^2 + \frac{4k^2}{n^2} \cdot \frac{q^2}{n^2}}},$$

and the *Magnification Factor* is therefore

$$\frac{A}{f/n^2} = \frac{1}{\sqrt{\left(1 - \frac{q^2}{n^2}\right)^2 + \frac{4k^2}{n^2} \cdot \frac{q^2}{n^2}}}.$$

For any given damping medium and elastic support, $\frac{4k^2}{n^2}$ will be a constant, say $= K^2$, so that the magnification factor depends only on the frequency ratio.

For $K = 0$, the expression coincides with the previous one for no damping as would be expected :

$$\frac{1}{\sqrt{\left(1 - \frac{q^2}{n^2}\right)^2}},$$

but as we can take either a +ve or a −ve value for the square root, we can plot the portion of the curve shown negative in Fig. 49 as positive as in Fig. 53*a*, and this figure also shows the variation of the magnification factor with frequency ratio for various values of K^2.

We see that

(*a*) When *q is small compared with n,* the magnification factor approaches unity, or the displacement of the load is very nearly the static displacement under the action of the disturbing force when not fluctuating.

(*b*) When *q is large compared with n,* the magnification factor approaches zero, i.e. the load remains almost stationary in space.

These results are the same as those already noted for no friction, as indicated here again for $K^2 = 0$, *hence for a disturbing* *force of frequency high or low compared with the natural, damping has very little effect on the motion whatever resistance is supplied.*

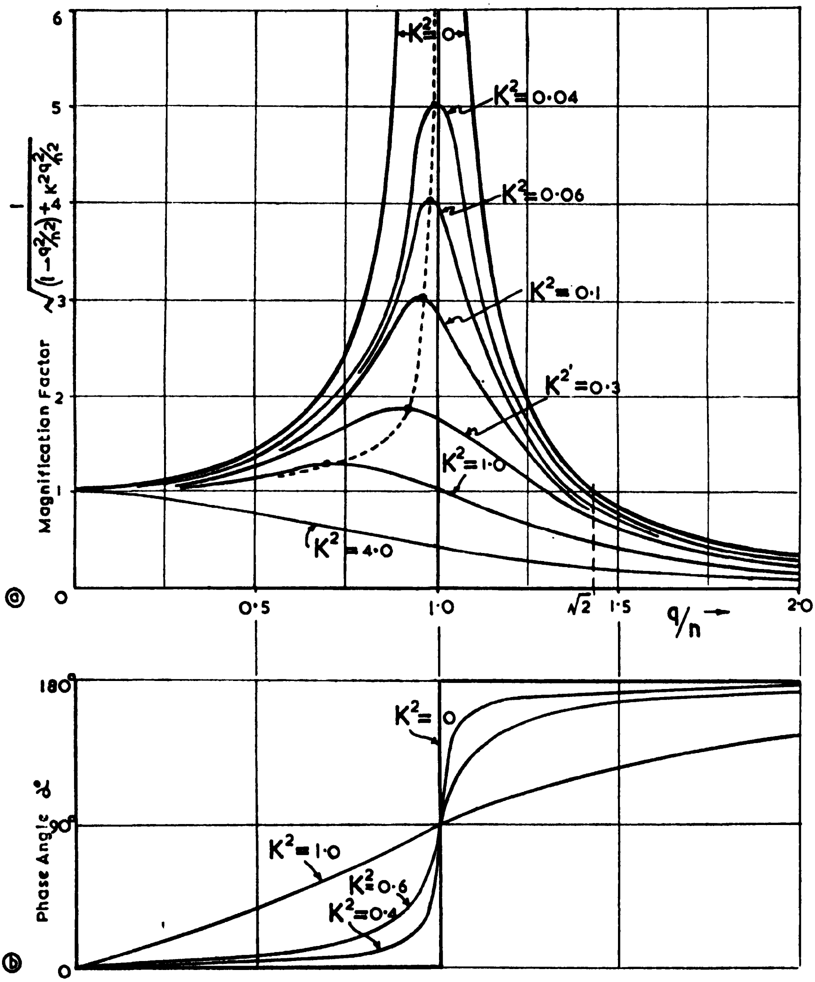


FIG. 53.

But in the neighbourhood of coincidence of the forced and natural frequencies (say $q/n = 0{\cdot}75 \rightarrow 1{\cdot}25$) damping produces a marked effect, the amplitude being kept finite on resonance. When $q = n$, the magnification factor $= 1/K$.

It will be noted, also, *that the amplitude becomes a maximum before exact coincidence* of q and n, but for small damping it is usual to take the maximum as occurring on coincidence.

We shall discuss this effect in a following section relating to the use of a damper to eliminate torsional oscillations of a shaft. We will, however, here determine the value of the frequency ratio at which the maximum amplitude occurs:

$$\text{Magnification factor} = \frac{1}{\sqrt{\left(1-\frac{q^2}{n^2}\right)^2 + K^2\frac{q^2}{n^2}}}.$$

For this to be a maximum, the denominator must be a minimum, hence differentiating this and equating to zero, we have, calling $(q/n)^2 = x$,

$$d/dx[(1-x)^2 + K^2x] = 0,$$

or

$$d/dx[1 + x^2 + (K^2 - 2)x] = 0\,;$$

$$\therefore\ 2x + (K^2 - 2) = 0,$$

or

$$x = (q/n)^2 = \frac{2-K^2}{2}.$$

This expression holds provided K^2 is less than 2 so that q/n is real.

Now in dealing with damped vibrations without disturbing forces we saw that, provided $n^2 > k^2$, a damped vibration of period $2\pi/\mu$ could occur where $\mu^2 = (n^2 - k^2)$, and this motion will be superimposed on the forced, which we are studying here.

Thus substituting $1 - \frac{1}{2}K^2$ for $\frac{q^2}{n^2}$ to obtain the maximum magnification factor, we have

$$\frac{1}{\sqrt{K^2\left(1-\frac{K^2}{4}\right)}}$$

$$= \frac{1}{\sqrt{\frac{4k^2}{n^2}\left(1-\frac{k^2}{n^2}\right)}} = \frac{1}{\sqrt{\frac{4k^2}{n^4}(n^2-k^2)}}$$

$$= \frac{n^2}{2k\mu},$$

or the maximum amplitude $=\frac{f}{n^2}\times$ magnification factor

$$=\frac{f}{2k\mu}.$$

When K^2 is greater than 2, the maximum occurs for $q/n=0$, being equal to the static deflexion P/F. The curves for such conditions are indicated on Fig. 53*a*, the amplitude always diminishing as q/n increases.

The variation of phase angle α with the frequency ratio is shown in Fig. 53*b*, and it will be seen that for *moderate* damping the conditions are not very different from those with no damping. For q/n small, α is approximately zero, and as q increases the increase in α is not great until we arrive in the region of resonance when the increase to 90 degrees becomes very rapid, and as we pass through these conditions the change of phase to approximately 180 degrees is also rapid.

For heavier damping, however, the increase in α from zero through 90 degrees and thence to 180 degrees is more gradual, being spread over the whole range, instead of being confined to the narrow margin around $q/n=$ unity.

Work done by a simple harmonic disturbing force. Consider, again, a damped forced vibration $x=A\cos qt$ in which the disturbing force vector P is at an angle α in advance of the displacement vector $oa'=A$. Let the force vector (Fig. 52*b*) be split up into two components

$P\cos\alpha$ parallel to the displacement vector,

$P\sin\alpha$ normal to the displacement vector.

Projecting, as before, on the vertical YY', it will be seen that the force component $(P\cos\alpha)\cos qt$ acts alternately for each $\frac{1}{4}$ cycle with and against the motion, so that the energy added in $\frac{1}{4}$ cycle is destroyed in the next, and the total energy added in one complete cycle is therefore zero.

The projection of the component $P\sin\alpha$, however, acts always in the direction of motion, continually adding energy to the system, which, however, is destroyed by damping, thus keeping the amplitude constant and finite (i.e. $P\sin\alpha=F_1qA$).

This is exactly similar to the disturbing force vector at resonance when $\alpha = 90$ degrees.

Measuring the angle from OY, the magnitude of the projection of the normal component will be $(P \sin \alpha) \cos (qt + 90°)$ or $-P \sin \alpha . \sin qt$, the negative sign showing that the vector is 90 degrees in advance of the displacement vector.

The work done during a small displacement δx is thus

$$(-P \sin \alpha . \sin qt)\delta x,$$

and for a whole cycle, $qt = 0 \rightarrow 2\pi$ radians,

$$\text{Work done} = \int_{qt=0}^{qt=2\pi} (-P \sin \alpha \sin qt)\, dx$$

$$= \int_{qt=0}^{qt=2\pi} (-P \sin \alpha \sin qt) \frac{dx}{dt} . dt,$$

and as $\dfrac{dx}{dt} = -Aq \sin qt$,

$$\text{Work done} = \int_{qt=0}^{qt=2\pi} (P \sin \alpha \sin qt) Aq \sin qt\, dt$$

$$= \int_{qt=0}^{qt=2\pi} (P \sin \alpha \sin qt) A \sin qt\, d(qt),$$

as we wish to integrate with regard to qt.

$$\therefore \text{ Work done} = P \sin \alpha . A \int_{qt=0}^{qt=2\pi} \sin^2 qt\, d(qt).$$

Now

$$\int \sin^2 qt\, d(qt) = \int \frac{1 - \cos 2qt}{2}\, d(qt) = \frac{qt}{2} - \frac{1}{4} \sin 2qt,$$

which reduces to π when the limits $0 \rightarrow 2\pi$ are imposed.

Thus

$$\text{Work done per cycle} = \underline{P \sin \alpha . A \times \pi,}$$

which is just sufficient to overcome damping and maintain the given amplitude.

It might interest the student to see how the above integral can be obtained from vector considerations. In (Fig. 52*b*), let the displacement vector oa' move through the small angle $\delta(qt)$ to oa'_1. Then $a'a'_1 = A . \delta(qt)$ and its projection gives the small displacement of the load δx, i.e.,

$$\delta x = A\, \delta(qt) \sin qt.$$

During this movement the force (i.e. vector $ON = P \sin \alpha \,.\, \sin qt$) may be considered as sensibly constant so that the work done is

$$ON \times \delta x = P \sin \alpha \, A \sin^2 qt \, \delta(qt).$$

Energy absorbed by damping. The damping force is

$$F_1 \frac{dx}{dt} = -F_1 A q \cdot \sin qt;$$

$$\therefore \text{ Work absorbed during one cycle} = \int_0^{2\pi} -F_1 A q \sin qt \,.\, dx$$

$$= -F_1 A q \int_0^{2\pi} \sin qt \frac{dx}{dt} \,.\, dt = +F_1 A^2 q^2 \int_{qt=0}^{qt=2\pi} \sin^2 qt \, dt$$

$$= \quad F_1 A^2 q \int_{qt=0}^{qt=2\pi} \sin^2 qt \, d(qt) = F_1 A^2 q \,.\, \pi.$$

If we equate this to the energy supplied we obtain

$$P \sin \alpha = F_1 A q,$$

a result which was obtained previously from our vectors.

Periodic displacement of the support. Let the support of an elastic structure, such as a spring, carrying a single load W, be given a simple harmonic motion of frequency $q/2\pi$ and amplitude a in a vertical plane as shown in Fig. 54.

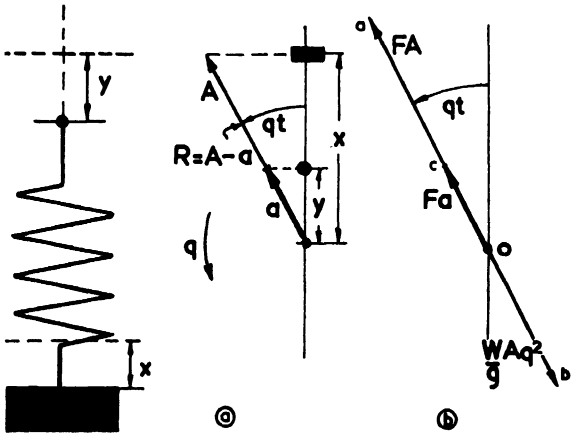


FIG. 54.

Let $y = a \cos qt =$ the displacement of the support from its mean position at the same instant that

$x =$ the displacement of the load W from its mean position, x and y being of the same sign when on the same side of their respective rest positions.

Then the alteration in length of the spring $= (x - y)$, and the equation of motion is

$$\frac{W}{g}\frac{d^2x}{dt^2} = -F(x-y),$$

or

$$\frac{d^2x}{dt^2} + n^2x = n^2y = n^2a \cos qt.$$

This equation is exactly similar to that above for a load subjected to a periodic disturbing force, Fa being substituted for P, or n^2a for f, and so the solution will be

$$x = A \cos qt = \frac{n^2a}{n^2-q^2} \,.\, \cos qt = \frac{n^2}{n^2-q^2} \,.\, y. \quad\text{(i)}$$

Thus the displacement of the load is an exact copy of that of the support magnified $\frac{n^2}{n^2-q^2}$ times.

Everything that has been said about simple harmonic forced vibrations applies to this motion, the only difference being that we are giving a periodic displacement to the support, instead of subjecting the load to a periodic disturbance with a fixed support.

Considering our previous vector solution, the vector representing the maximum value of the disturbing force P now represents Fa, and it generates the simple harmonic motion of the support to the same scale that the vector oa generates the displacement of the load, since the projection of this latter vector is $FA \cos qt$, i.e. $F \,.\, x$, where A is the amplitude of the load and here is equal to $\frac{n^2a}{n^2-q^2}$. As before, consider the solution (i) and its vector counterpart (Fig. 54) :

$\underline{q < n}$, or $ob < oa$, x and y are of the same sign, i.e. the load and the support move *in the same direction*, the two vectors being in phase. Note the difference between this case and the corresponding forced vibration—when the vector generating the disturbing force is in phase with the displacement vector its projection, which gives the instantaneous value of the force, is not necessarily in the same direction as the motion of the load, but changes each ¼ revolution. When q is small compared with n the magnification factor is almost unity, i.e. the motion of the load is almost the same as that of the support, the whole system moving as a solid mass.

$q=n$, the solution fails as above, and we should now obtain

$$x = \frac{na \cdot t}{2} \cdot \sin qt \rightarrow \infty \quad \left[\text{cf. } \frac{ft}{2q} \cdot \sin qt\right].$$

The displacement of the load is now 90 degrees behind that of the support, i.e. the support is passing through its mean position as the load is at its maximum position.

$q>n$, $ob>oa$, x and y are now of opposite sign, i.e. the load moves up as the support moves down and vice versa, the vectors being 180 degrees out of phase.

If the frequency of the support is very high compared with the natural frequency then the vibration of the support will hardly be communicated to the load *which will remain almost stationary in space.*

We might anticipate these results from a consideration of the inertia of the load, time being required for a change in position to be caused. Hence the slower the motion of the support the more nearly will the load follow it, but when the oscillations of the support become very rapid the load will not have time to make any appreciable alteration in its position.

We may illustrate these phenomena very easily by means of two simple pendulums arranged as shown in Fig. 55. OX is a light rod pivoted at O and carrying a heavy weight X, the position of which can be altered at will. From an arm projecting from this rod hangs a simple pendulum of constant length L.

By setting the pendulum OX in vibration we shall give approximately a simple harmonic motion to the support of the second pendulum. If now the length OX is made greater than L, that is, the frequency of the support is less than the natural frequency, both bobs will be seen to move together in the same direction reaching their maximum positions together.

If $OX = L$, the amplitude of the attached pendulum tends to infinity, and it will be noticed that as the bob swings with increasing amplitude its support, i.e. pendulum OX, is passing its ***mean*** position as the bob arrives at its ***maximum*** displacement.

If OX is made less than L, the frequency of the support is now greater than the natural and it will be found that the supporting pendulum will arrive at its maximum displacement on one side as the other attains its maximum on the *opposite* side.

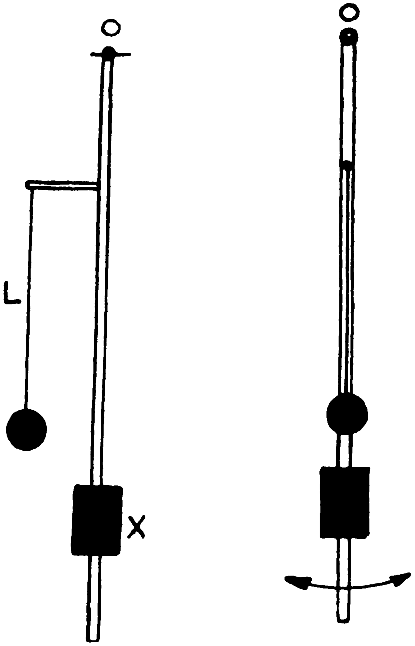


FIG. 55.

This system is, of course, similar to that of the double pendulum in which W_1 is heavy compared with W_2 (page 67).

For the purpose of this experiment we have assumed that the pendulum OX was kept vibrating by a series of taps so that the support of the attached pendulum had a S.H.M. of constant amplitude imparted to it. If, however, the system is started and left to its own devices and the lengths of the two pendulums are approximately the same, then the phenomenon, mentioned in the discussion of the double pendulum, of transference of energy from one to the other can again be strikingly illustrated.

Let us examine mathematically the case of a simple pendulum whose support is given a S.H.M. (Fig. 56*a*).

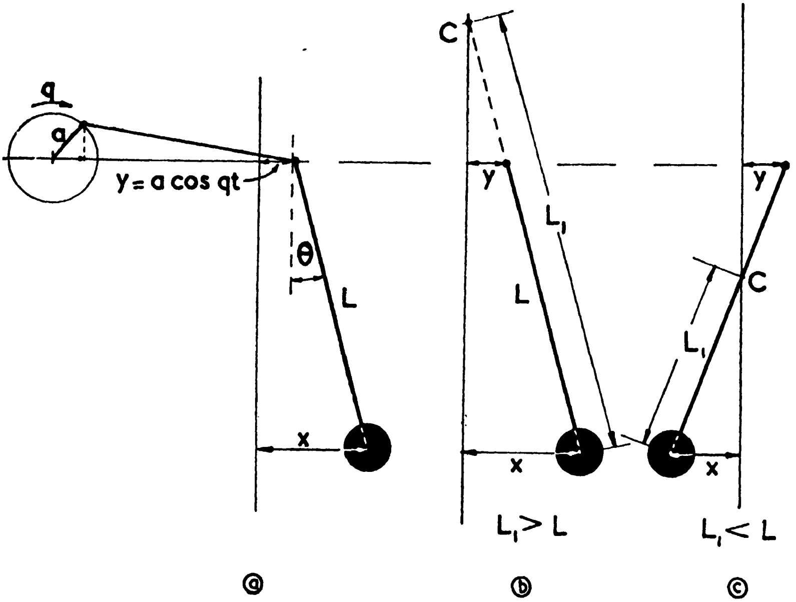


FIG. 56.

It will be remembered that the period of a simple pendulum of length L is

$$2\pi\sqrt{\frac{L}{g}},$$

hence $$n^2 = g/L.$$

Let x = the displacement of the bob from its mean position ;
$y = a\cos qt$ = the displacement of the support from its mean position at the same instant, x and y being of the same sign when on the same side of the mean, so that the actual displacement of the bob from the vertical $= (x - y)$.

Hence the restoring torque $= -W(x-y)$ and the equation of motion will be

$$\frac{W}{g} \cdot L^2 \cdot \frac{d^2\theta}{dt^2} = -W(x-y).$$

Now the linear acceleration of the bob in a direction tangential to the arc of its swing is $\frac{d^2\theta}{dt^2} \cdot L$ and, making the usual assumption that θ is small, we shall not be far wrong in assuming that this acceleration acts along the horizontal, in other words

$$\frac{d^2x}{dt^2}=\frac{d^2\theta}{dt^2} \cdot L\,;$$

$$\therefore\ \frac{W}{g} \cdot L \cdot \frac{d^2x}{dt^2} = -W(x-y),$$

or

$$\frac{d^2x}{dt^2}+n^2x=n^2y=n^2a\cos qt.$$

Thus the solution is

$$x=\frac{n^2a}{n^2-q^2}\cos qt$$

$$=\frac{n^2}{n^2-q^2} \cdot y,$$

the result being exactly similar to that of the spring above, as we should expect.

If we make $q^2=g/L_1$, i.e. the frequency of the support is the same as that of a simple pendulum of length L_1, then the vibration becomes

$$x=\frac{g/L}{g/L-g/L_1} \cdot y$$

$$=\frac{L_1}{L_1-L} \cdot y,$$

or

$$\frac{x}{y}=\frac{L_1}{L_1-L}.$$

This result can be represented by the diagrams shown in Fig. 56*b* and *c*, which give the configurations for q both less and greater than n, i.e. L_1 greater and less than L.

The pendulum moves, in fact, as if it was supported from the point C and was of length L_1.

Relative amplitude. The *relative displacement* of the load to its support (Fig. 54*a*) is $x-y$, since the vectors generating the motion of the load and of the support are always in phase or 180 degrees out of phase (except when $q=n$).

The Relative Amplitude R is given by the difference of these two vectors which is $A-a$ and the projection of this vector length gives $x-y$.

Thus
$$R = A - a = \left[\frac{n^2}{n^2-q^2} - 1\right] a = \frac{q^2}{n^2-q^2} \cdot a$$
$$= \frac{q^2/n^2}{1-q^2/n^2} \cdot a.$$

The curve (Fig. 57) shows the variation of relative amplitude

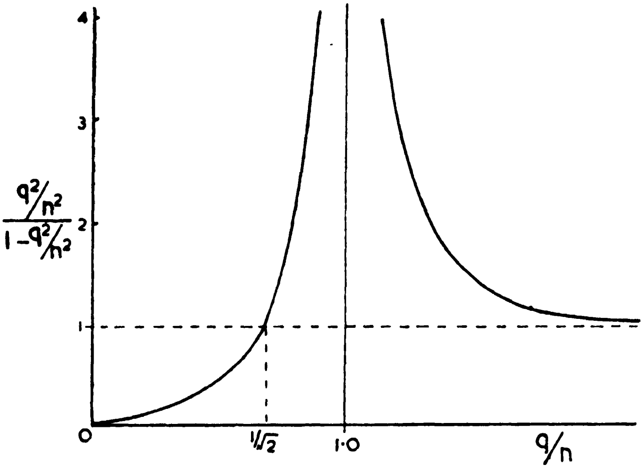


FIG. 57.

with the frequency ratio q/n, and the difference between this and magnification factor should be noted.

When $q/n = 0$ $(x-y) = 0$

$q/n = 1$ $(x-y) = \infty$

q/n very great $(x-y) \rightarrow y$.

Periodic displacement of support, with damping varying as the velocity of the load, i.e. equivalent to a dashpot fixed between the load and the ground (Fig. 58*a*). Our equation of motion is now

$$\frac{W}{g}\frac{d^2x}{dt^2} + F_1\frac{dx}{dt} + F(x-y) = 0,$$

or
$$\frac{W}{g}\frac{d^2x}{dt^2}+F_1\frac{dx}{dt}+Fx=Fy=Fa\cos qt,$$

where, as before, $y=a\cos qt$ is the simple harmonic motion of the support.

This is exactly similar to our previous equation for a forced damped vibration except that Fa is substituted for P, or n^2a for f, as with the undamped motion.

The same vector solution may be adopted from which it will be seen that the vector generating the displacement of load is at an angle α behind the vector generating the displacement of the support (which was previously the disturbing force vector, but now its length is Fa and its projection $Fa\cos qt$, i.e. Fy) (Fig. 58*b*).

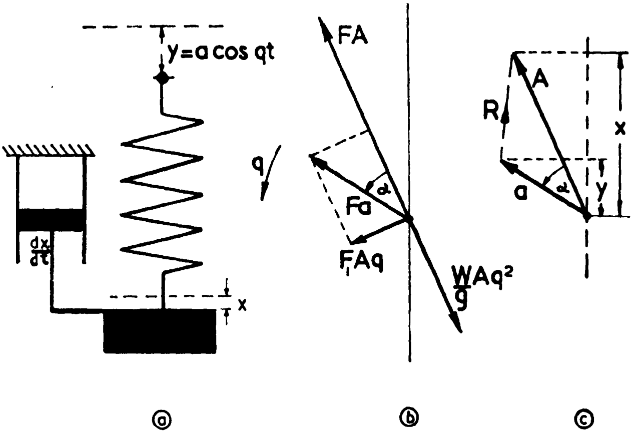


FIG. 58.

Thus the solution is

$$x=A\cos(qt-\alpha),$$

and the amplitude A of the load is given by

$$\frac{n^2a}{n^2-q^2}\cdot\cos\alpha,\quad\text{or}\quad\frac{n^2a}{2kq}\sin\alpha,\quad\text{or}\quad\frac{n^2a}{\sqrt{(n^2-q^2)^2+4k^2q^2}},$$

where
$$\tan\alpha=\frac{2kq}{n^2-q^2},\ \text{etc.},$$

all these expressions being obtained in exactly the same way as before.

Since the vectors which generate the motion of the load and of the support are at the phase angle α, they do not reach their maximum together and the relative amplitude R is their vector difference.

This vector is revolving with angular velocity q and its projection gives the relative displacement of load to support, i.e. $x-y$ at any instant (Fig. 58*c*).

Periodic displacement of the support with damping varying as the relative velocity between the load and the support. In this case the effect is equivalent to a dashpot which is fitted between the load and its support, e.g. the shock absorbers of a car, the frictional resistance depending on the *relative* velocity between the piston and its cylinder, i.e. $dx/dt - dy/dt$ (Fig. 59).

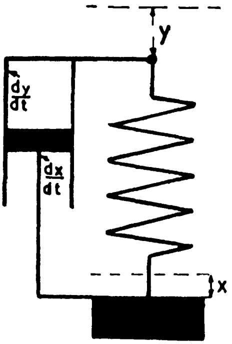


FIG. 59.

Thus the equation of motion becomes

$$\frac{W}{g}\frac{d^2x}{dt^2}+F_1\left(\frac{dx}{dt}-\frac{dy}{dt}\right)+F(x-y)=0.$$

Now
$$y=a\cos qt;$$
$$\therefore\ dy/dt=-aq\sin qt,$$
so that
$$\frac{W}{g}\frac{d^2x}{dt^2}+F_1\frac{dx}{dt}+Fx=(Fa)\cos qt-(F_1aq)\sin qt$$
$$=\sqrt{(Fa)^2+(F_1aq)^2}\cos(qt+e),$$

by conversion of the right-hand side into the alternative cosine form, where

$$\tan e = \frac{F_1 aq}{Fa}$$

$$= \frac{2kq}{n^2} = K \cdot \frac{q}{n}.$$

The motion is therefore exactly the same as if a disturbing force of maximum value $P = \sqrt{(Fa)^2 + (F_1 aq)^2}$ was acting, i.e.

$$x = A \cos[(qt + e) - \alpha],$$

and the amplitude

$$A = \frac{Pg/W}{n^2 - q^2} \cdot \cos\alpha$$

$$= \frac{Pg/W}{\sqrt{(n^2 - q^2)^2 + 4k^2q^2}},$$

as for a forced vibration, where P has the meaning given above.

We may represent both sides of the equation of motion by vectors, and in Fig. 60*a*, *b*, the vector P has been drawn at an

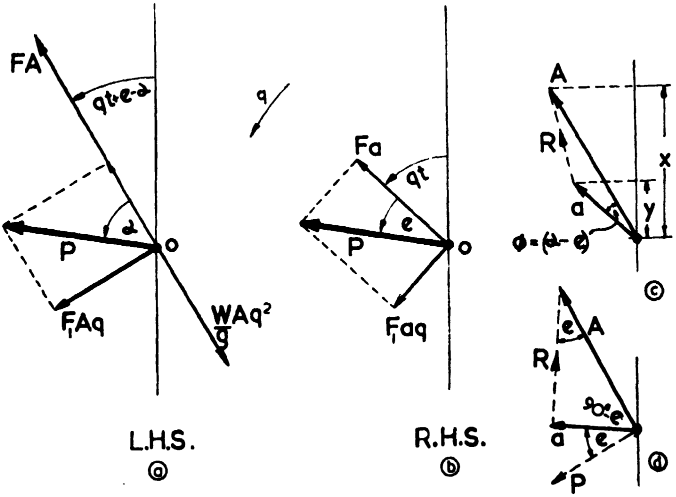


FIG. 60.

angle $(qt+e)$ to the projection line so that the " disturbing force " is $P\cos(qt+e)$ and the forced motion is

$$x=A\cos(qt+e-\alpha)$$

lagging behind by the phase angle α.

Fig. 60*a* shows the vectors for the left-hand side of the equation and is exactly the same as for a forced damped vibration, while (*b*) shows the vectors for the terms on the right-hand side, i.e. the two components Fa and F_1aq making up the vector P.

The vector generating the motion of the load (of length A) is in the same direction as vector FA (l.h.s.) as usual, and that generating the motion of the support (of length a) is in the same direction as Fa (r.h.s.) (Fig. 60*c*), and the phase angle ϕ between them is $(\alpha-e)$.

The relative amplitude R is their vector difference.

We can by a little juggling obtain expressions for ϕ, A, and R in terms of the frequency ratio q/n, e.g.

$$\cos\phi=\cos(\alpha-e)=\cos\alpha\cos e-\sin\alpha\sin e$$
$$=\cos\alpha\cos e(1+\tan\alpha\tan e),$$

$$\tan\alpha=\frac{2kq}{n^2-q^2}=\frac{K.\lambda}{1-\lambda^2}$$

where, for convenience, $\lambda=q/n$;

$$\tan e=\frac{2kq}{n^2}=\frac{K.q}{n}=K.\lambda.$$

Since $\dfrac{1}{\cos^2\alpha}=1+\tan^2\alpha$, we have (see also page 158),

$$\cos\alpha=\frac{n^2-q^2}{\sqrt{(n^2-q^2)^2+4k^2q^2}}=\frac{1-\lambda^2}{\sqrt{(1-\lambda^2)^2+K^2\lambda^2}}.$$

Similarly

$$\frac{1}{\cos^2 e}=1+\tan^2 e=1+K^2\lambda^2,$$

or

$$\cos e=\frac{1}{\sqrt{1+K^2\lambda^2}};$$

$$\therefore\ \cos\phi=\frac{1-\lambda^2}{\sqrt{(1-\lambda^2)^2+K^2\lambda^2}}\cdot\frac{1}{\sqrt{1+K^2\lambda^2}}\left[1+\frac{K\lambda}{1-\lambda^2}\cdot K\lambda\right],$$

which reduces to

$$\cos\phi = \frac{1-\lambda^2+K^2\lambda^2}{\sqrt{(1-\lambda^2)^2+K^2\lambda^2}\,.\,\sqrt{1+K^2\lambda^2}}\,.$$

Again, since

$$A = \frac{Pg/W}{\sqrt{(n^2-q^2)^2+4k^2q^2}} = \frac{\frac{g}{W}\,.\,a\sqrt{F^2+F_1{}^2q^2}}{\sqrt{(n^2-q^2)^2+4k^2q^2}}$$

$$= \frac{a\sqrt{(n^2)^2+4k^2q^2}}{\sqrt{(n^2-q^2)^2+4k^2q^2}} = \frac{a\sqrt{1+K^2\,.\,q^2/n^2}}{\sqrt{(1-q^2/n^2)^2+K^2\,.\,q^2/n^2}}$$

$$= a\,.\,\frac{\sqrt{1+\lambda^2K^2}}{\sqrt{(1-\lambda^2)^2+K^2\lambda^2}}\,.$$

Now for the relative amplitude R, we have

$$R^2 = A^2 + a^2 - 2aA\cos\phi,$$

and, substituting the values obtained above, this will reduce to

$$R^2 = a^2\,.\,\frac{(\lambda^2)^2}{(1-\lambda^2)^2+K^2\lambda^2},$$

or

$$R = a\,.\,\frac{\lambda^2}{\sqrt{(1-\lambda^2)^2+K^2\lambda^2}}\,.$$

When

$$\lambda = 1 \quad \text{or} \quad q = n,$$

$$\alpha = 90^\circ \quad \text{as usual,}$$

$$\tan e = K = 2k/n,$$

and

$$R = \frac{a}{K} = a\cot e.$$

Since $\phi = (\alpha - e)$ and is now $90^\circ - e$, this means that R is perpendicular to vector a, as in Fig. 60*d*, and therefore $R^2 = A^2 - a^2$.

This result can be obtained in a variety of ways, e.g.:

when

$$\alpha = 90^\circ,\ P = F_1Aq,$$

and

$$\therefore\ (F_1Aq)^2 = (Fa)^2 + (F_1aq)^2,$$

which reduces to

$$A^2 = a^2\,.\,\frac{1+K^2}{K^2} = a^2\,.\,\frac{1}{\cos^2 e}\times\frac{\cos^2 e}{\sin^2 e},$$

or

$$\underline{a = A\sin e.}$$

Periodic displacement of support—magnification factor or dynamic magnifier. We have seen that, with no damping, the amplitude A of the load is $\frac{n^2}{n^2-q^2}$ times the amplitude a of the support.

$$\text{Thus the dynamic magnifier} = \frac{\text{amplitude of load}}{\text{amplitude of support}} = \frac{A}{a}$$

$$= \frac{n^2}{n^2 - q^2}.$$

This is exactly the same as for a periodic disturbing force, for which case it will be remembered that the

$$\text{dynamic magnifier} = \frac{\text{amplitude of load}}{\text{static displacement } P/F}$$

$$= \frac{\text{maximum force in structure}}{\text{maximum disturbing force}}.$$

Referring to the two further cases including damping which have been discussed in the previous sections we see that:

(1) With damping varying as the velocity of the load,

$$A = \frac{n^2 a}{\sqrt{(n^2 - q^2)^2 + 4k^2q^2}};$$

$$\therefore \text{ dynamic magnifier} = \frac{A}{a} = \frac{n^2}{\sqrt{(n^2 - q^2)^2 + 4k^2q^2}},$$

as we should expect.

And

(2) With damping varying as the relative velocity between load and support,

$$\frac{A}{a} = \frac{\sqrt{(n^2)^2 + 4k^2q^2}}{\sqrt{(n^2 - q^2)^2 + 4k^2q^2}} \quad \text{[page 175].}$$

Variable maximum value of disturbing force. There are two important cases in which the maximum value P of the disturbing force varies as the square of the frequency instead of being constant:

(1) Due to the centrifugal reaction of an unbalanced rotating mass w at radius r. At any speed ω, this reaction is $\frac{w}{g}\omega^2 r$ and the component of this force in any given direction in the plane of rotation will be simple harmonic of the form $\frac{w}{g}\omega^2 r \,.\, \cos \omega t$.

(2) Due to the inertia reaction of a reciprocating mass w, the force set up in the line of stroke, assuming an infinitely long connecting rod, is $\frac{w}{g}\omega^2 r \, . \cos \omega t$ (see page 214).

Thus $P = \frac{w}{g}\omega^2 r$ in either case, and when this disturbing force acts on a single mass of weight W, elastically mounted,

$$x = \frac{Pg/W}{n^2 - \omega^2} \cdot \cos \omega t,$$

ω being equivalent to q in our previous discussion.

Thus $$x = \frac{w}{W} r \left[\frac{\omega^2/n^2}{1 - \omega^2/n^2}\right] \cos \omega t.$$

The variation of this displacement with frequency ratio is similar to that of Fig. 57, and the motion is equal to the *relative displacement* between the load and its support if the latter was given a simple harmonic motion $\frac{w}{W} r \cos \omega t$.

Constant disturbing force. In certain cases an elastically mounted mass may be subjected to a disturbing force which is constant both in magnitude and direction.

It should not be difficult for the student to visualise that such a force will merely displace the body to a new rest position, about which free natural vibrations can occur ; but it may be of interest to deduce this mathematically.

The equation of motion will be

$$\frac{W}{g} \frac{d^2x}{dt^2} + Fx = P,$$

or $$\frac{d^2x}{dt^2} + n^2 x = f,$$

where $$\frac{Pg}{W} = f.$$

We may write this as

$$\frac{d^2x}{dt^2} + n^2\left(x - \frac{f}{n^2}\right) = 0,$$

and since the differential of a constant (here f/n^2) is zero, we may further adjust the above to the proper simple harmonic form, i.e.

$$\frac{d^2}{dt^2}\left(x-\frac{f}{n^2}\right)+n^2\left(x-\frac{f}{n^2}\right)=0,$$

which represents the free natural vibrations of frequency $\frac{n}{2\pi}$ about a rest position at a distance $\frac{f}{n^2}$ from the original from which x was measured, and $\frac{f}{n^2}=\frac{P}{F}$, i.e. the static displacement due to the force P.

Forced torsional oscillations. Although in the previous sections we have dealt with systems in linear motion, similar equations will, of course, apply for a load of moment of inertia I (absolute units) which can perform angular oscillations, controlled by the torsion of the elastic support, and subjected to a periodic disturbing torque $T \cos qt$.

For instance, the equation of motion of such a system including damping, would be

$$\frac{I}{g}\frac{d^2\theta}{dt^2}+T_1\frac{d\theta}{dt}+T_0\theta=T\cos qt,$$

where T_1 = the damping torque per unit angular velocity ;

T_0 = the stiffness of elastic support ;

T = the maximum value of the simple harmonic disturbing torque.

Exactly similar reasoning will apply to these systems as for linear systems both in regard to a disturbing torque or a periodic angular oscillation given to the support.

CHAPTER VII

FORCED VIBRATIONS OF MULTIPLE SYSTEMS. SINGLE DISTURBING FORCE.

(*a*) **No damping.** We have seen that if a simple harmonic disturbing force acts on a single load system, a vibration of frequency equal to that of the force is set up. It is reasonable to assume, therefore, that the same is true for a multiple system.

The method of solution affords no difficulty—we write down the equations of motion of the loads as before and introduce the disturbing force at the required point.

As an example, consider the two-load system (Fig. 61) in which a disturbing force $P \cos qt$ acts on W_1:

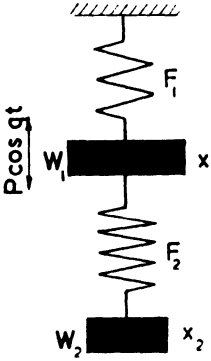


FIG 61.

Let x_1, x_2 be the displacements of the loads at the same instant from their rest positions, so that the equations of motion will be

$$\frac{W_1}{g}\frac{d^2x_1}{dt^2} = -F_1x_1 - F_2(x_1 - x_2) + P \cos qt,$$

$$\frac{W_2}{g}\frac{d^2x_2}{dt^2} = -F_2(x_2 - x_1).$$

Now, *without damping*, all the loads will vibrate in phase (or 180 degrees out) with the disturbing force, i.e. we may write

$$x_1 = y_1 \cos qt, \quad \therefore \ \frac{d^2x_1}{dt^2} = -q^2y_1 \cos qt = -q^2x_1;$$

$$x_2 = y_2 \cos qt, \quad \therefore \ \frac{d^2x_2}{dt^2} = -q^2y_2 \cos qt = -q^2x_2.$$

It will be more convenient here to substitute the first expression for the acceleration, $-q^2y\cos qt$, rather than the second, as we can then eliminate the cosine term and convert the load displacements into amplitudes y_1, y_2 which it is our object to determine. Thus:

$$(1)\quad \frac{W_1}{g}q^2y_1 = +F_1y_1 + F_2(y_1 - y_2) - P,$$

$$(2)\quad \frac{W_2}{g}q^2y_2 = F_2(y_2 - y_1).$$

From (2) $$y_2\left(\frac{W_2}{g}q^2 - F_2\right) = -F_2y_1.$$

Substituting in (1),

$$y_1\left[\frac{W_1}{g}q^2 - F_1 - F_2 - \frac{F_2^2}{\frac{W_2}{g}q^2 - F_2}\right] = -P;$$

$$\therefore\ y_1 = \frac{-P\left(\frac{W_2}{g}q^2 - F_2\right)}{\left[\frac{W_1}{g}q^2 - (F_1 + F_2)\right]\left(\frac{W_2}{g}q^2 - F_2\right) - F_2^2},$$

and $$y_2 = \frac{+F_2P}{\text{denominator as for } y_1}.$$

It will be seen that both amplitudes will become infinite together for values of q obtained by equating the denominator to zero, i.e. when

$$\left[\frac{W_1}{g}q^2 - (F_1 + F_2)\right]\left(\frac{W_2}{g}q^2 - F_2\right) - F_2^2 = 0,$$

or $$\frac{W_1}{g}\cdot\frac{W_2}{g}(q^2)^2 - q^2\left[\frac{W_1}{g}\cdot F_2 + \frac{W_2}{g}(F_1 + F_2)\right] + F_1F_2 = 0.$$

If we solve the same problem for the natural frequencies $n/2\pi$, exactly the same equation as above will be obtained except that n replaces q. This should be expected since it merely means that the amplitudes become infinity when $q = n$, i.e. upon resonance.

The student should solve the same problem but apply the disturbing force at the second load.

For a large number of loads a modification of the tabular method of solution is convenient and an example has been worked out on page 190 in the section concerned with torsional oscillations.

Note that we solve for the amplitudes of the loads which are finite for all values of q except for resonant conditions ; and if the amplitude ratio for consecutive loads is negative a node appears in the elastic connexion. The position and number of nodes will vary with the value of q.

The free vibrations, which can exist independently of the forced, would have to be superimposed on the forced motion to obtain the resultant motion, but damping will in time destroy them.

Periodic displacement of support—multiple systems. If instead of a disturbing force, the support of the system shown in Fig. 61 is given a simple harmonic motion $y=a\cos qt$, the equations of motion would be

$$(1)\quad \frac{W_1}{g}\frac{d^2x_1}{dt^2} = -F_1(x_1-y)-F_2(x_1-x_2),$$

or

$$\frac{W_1}{g}\frac{d^2x_1}{dt^2} = -F_1x_1-F_2(x_1-x_2)+F_1a\cos qt,$$

and

$$(2)\quad \frac{W_2}{g}\frac{d^2x_2}{dt^2} = -F_2(x_2-x_1),$$

the equations being exactly as if a disturbing force of maximum value $P=F_1a$ was applied to the load.

No particular difficulty should be experienced with similar problems, but when damping is introduced causing phase differences between the loads and the disturbing force this method is not easily applicable.

The mode of attack which we shall now adopt consists in representing each component of the equations of motion as a vector by the use of " complex quantities ".

We have already solved the case of a single load graphically by means of vectors, but now it will be necessary to represent these vectors mathematically.

VECTORS

If V represents the vector OA (Fig. 62), then $-V$ represents the vector AO, i.e. the effect of the factor -1 is to reverse the direction of, or, what is equivalent, to rotate the vector through 180° or π radians in the positive direction (anticlockwise). Similarly $-1 \times -V$ or $(-1)^2V$ means $-V$ rotated through π radians or V rotated through 2π radians, and therefore $(-1)^nV$ represents a rotation of V through $n\pi$ radians, which holds whether n is positive, negative, integral or fractional.

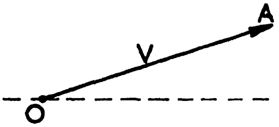


FIG. 62.

Thus $(-1)^n$ *always* means a rotation through $n\pi$ radians.

If we make $n=\theta/\pi$ we see that the factor $(-1)^{\theta/\pi}$ rotates any vector through an angle θ.

Of particular importance is the factor $(-1)^{\frac{1}{2}}$ *or* $\sqrt{-1}$ *generally denoted by the symbol* i *or* j, *which thus represents a rotation of* $\pi/2$ *or* 90°.

Hence iV *represents a rotation of a vector* V *through* 90° *in the positive direction.*

Consider Fig. 63: if $V_\theta=OB$ represents the vector V rotated through the angle θ, then, drawing BC perpendicular to OA, OC denotes the vector $V \cos\theta$, being in the same direction OA as the original vector, and CB denotes a vector of magnitude $V\sin\theta$, but being turned through 90 degrees from the original direction OA, it will be represented as a vector by

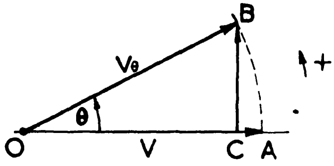


FIG. 63.

$$iV\sin\theta.$$

But vector $OB=$ vector $OC+$ vector CB,

i.e. $$V_\theta=\cos\theta\,.\,V+i\sin\theta\,.\,V$$
$$=(\cos\theta+i\sin\theta)\,V,$$

and we have already shown that

$$V_\theta=(-1)^{\theta/\pi}\,.\,V.$$

Hence $(-1)^{\theta/\pi}$ *and* $(\cos\theta+i\sin\theta)$ *are equal and both represent a rotation of a unit vector through an angle* θ.

Now it can also be shown by trigonometry that

$$\cos\theta + i\sin\theta = e^{i\theta} \quad \text{.......................(1)}$$

and ∴ writing $-\theta$ for θ,

$$\cos\theta - i\sin\theta = e^{-i\theta}. \quad \text{....................(2)}$$

Thus $e^{i\theta}$ *also represents the rotation of a unit vector through an angle* θ, *and similarly* $e^{-i\theta}$ *a rotation in the reverse direction.**

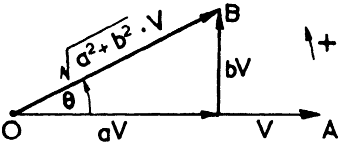


FIG. 64.

It should therefore be clear that an expression $(a+ib)V$ represents a vector with a component aV along the original vector V, and a component bV at right angles, i.e. a vector of magnitude $\sqrt{a^2+b^2}\,.\,V$ and making an angle of tangent $\frac{b}{a}$ with V in the positive direction of rotation, i.e. in advance of V (Fig. 64).

---

* [Whilst on the subject of vectors, it is interesting to note the vector meaning of the exponential values of the sine and cosine, e.g. adding expressions (1) and (2) we get:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

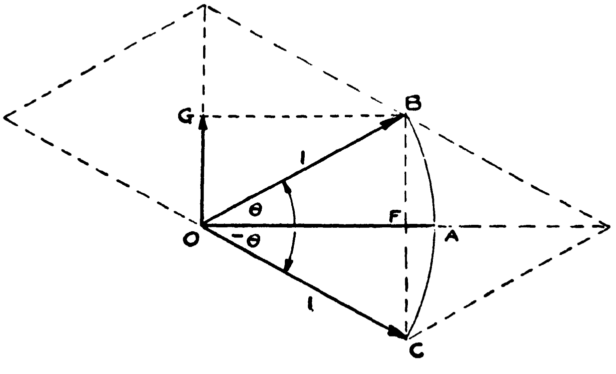
subtracting

$$\sin\theta = -\frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Now $e^{i\theta}$ = rotation of a *unit* vector through an angle θ, i.e. OA to OB. Similarly, $OC = e^{-i\theta}$.

Then the vector sum of OB and OC divided by 2 is $OF = OB\cos\theta = \cos\theta$.

Similarly the ½ vector difference of OB and OC is $OG = OB\sin\theta = \sin\theta$, but being at right angles to the original direction OA is thus completely specified by $i\sin\theta$.]



Similarly $(a-ib)V$ represents a component along V and a right-angle component in the reverse direction so that the vector lags behind V by an angle of tangent $-\frac{b}{a}$.

The factor a is called the real or axial component and b the unreal or non-axial component, the whole expression being termed a " complex quantity ".

Consider again our vector solution for a forced simple harmonic oscillation of a single load with damping (Fig. 65*a*).

The vector representing the maximum value of the force to overcome friction is at right angles to the vectors *oa* and *ob*, and as a vector equation we may therefore write

$$\text{vector } oc + \text{vector } ce = \text{vector } oe,$$

i.e.

$$\left(FA - \frac{W}{g}q^2A\right) + i \, . \, F_1qA = P,$$

and therefore

$$A = \frac{P}{F - \frac{W}{g}q^2 + iF_1q} = \frac{P}{a+ib},$$

or

$$P = (a+ib)A.$$

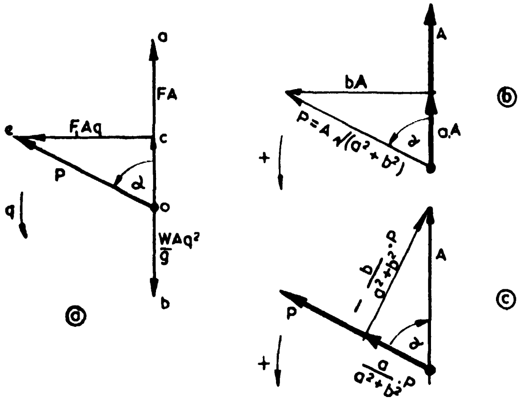


FIG. 65.

From the above, this represents a vector of magnitude $\sqrt{a^2+b^2} \, . \, A$ at an angle α in advance of the displacement vector A, which is taken as datum, where

$$\tan\alpha = \frac{b}{a} = \frac{F_1q}{F - \frac{W}{g}q^2}, \quad \text{(Fig. 65}b\text{)}$$

or, alternatively, measuring rotation from the vector P instead of from A,

$$A = \frac{P}{a+ib}.$$

Multiplying top and bottom by $(a-ib)$, and remembering that $(i)^2 = -1$, we get the result in the required form

$$A = \frac{P}{a+ib} \times \frac{(a-ib)}{(a-ib)}$$

$$= \frac{P}{a^2+b^2}(a-ib),$$

which represents a vector of magnitude

$$\frac{P}{a^2+b^2} \times \sqrt{a^2+b^2}, \quad \text{i.e.} \quad \frac{P}{\sqrt{a^2+b^2}},$$

having a component $\frac{a \,.\, P}{a^2+b^2}$ along P, and a right-angle component $-\frac{b \,.\, P}{a^2+b^2}$ (Fig. 65*c*).

From the above discussion it will be realised that any simple harmonic function such as a disturbing force previously considered as $P \cos qt$, which is generated by a vector of magnitude P rotating with angular velocity q, may be represented vectorially as

$$P \,.\, e^{iqt}.$$

If this force acts on an elastic system a vibration of the same frequency will be produced, the generating vector of which may lag behind that of the disturbing force by an angle α, i.e., as previously considered,

$$x = A \cos (qt - \alpha),$$

where A is the amplitude of the motion.

This can now be represented vectorially by

$$x = A e^{i(qt-\alpha)}$$

$$= A e^{-i\alpha} \,.\, e^{iqt} = Z \,.\, e^{iqt},$$

where $Z = A e^{-i\alpha}$ merely means that the vector A has been turned through an angle $-\alpha$ from the disturbing force vector, and e^{iqt} shows that this vector is rotating with angular velocity q.

Thus whether there is a phase difference or not, the displacement can always be represented vectorially as above, the constant Z, if not complex, being equal to the amplitude A (i.e.

there is no phase difference as occurs without damping) *while, if complex, can always be put into the form*

$$Z=(\pm a-ib)P=Ae^{-i\alpha},$$

taking the disturbing force vector P as datum, so that the magnitude of the vector represented (i.e. A) is $\sqrt{a^2+b^2}\,.\,P$, and the angle of lag $(-\alpha)$ has a tangent $-\frac{b}{a}$. The quantity a can be positive or negative so that the vector A can only be situated in the third or fourth quadrants measured in the positive or anticlockwise direction from the disturbing force vector P, i.e. the angle of *lag* is between 0° and 180°.

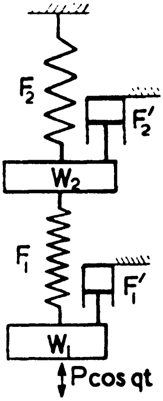


FIG. 66.

In order to illustrate the method, let us take the case of two loads and let different amounts of damping act on each load as indicated in Fig. 66, the resistance being F_1' and F_2' per unit velocity respectively. A disturbing force $P\cos qt$ acts on W_1.

Our equations of motion will be :

$$(1)\quad \frac{W_1}{g}\frac{d^2x_1}{dt^2}+F_1'\frac{dx_1}{dt}+F_1(x_1-x_2)=P\cos qt;$$

$$(2)\quad \frac{W_2}{g}\frac{d^2x_2}{dt^2}+F_2'\frac{dx_2}{dt}+F_1(x_2-x_1)+F_2x_2=0.$$

Now the frequency of vibration of the system may be assumed to be that of the disturbing force $q/2\pi$, but due to the different dampings the motions of the loads will be of different phase.

Thus, representing the disturbing force as a vector by $P \,.\, e^{iqt}$, let the displacements be similarly denoted as

$$x_1 = y_1 e^{i(qt-\alpha)} = A e^{iqt}, \qquad \text{where } A = y_1 e^{-i\alpha}$$

$$x_2 = y_2 e^{i(qt-\beta)} = B e^{iqt}. \qquad B = y_2 e^{-i\beta}$$

Then, differentiating,

$$\frac{dx_1}{dt} = i \,.\, qA e^{iqt},$$

$$\frac{d^2x_1}{dt^2} = -q^2 A e^{iqt},$$

and similarly for x_2.

These expressions represent vectors at 90° and 180° to their respective displacement vectors as we should anticipate.

Rewriting the original equations in their vector form, we therefore obtain upon substituting the above values :

$$\text{(i)} \quad -\frac{W_1}{g} q^2 A + iF_1'qA + F_1(A-B) = P,$$

$$\text{(ii)} \quad -\frac{W_2}{g} q^2 B + iF_2'qB + F_1(B-A) + F_2 B = 0.$$

Let us assume simple numerical values for the various quantities making the frictional coefficients different—say :

$F_1 = F_2 = 1$ lb. per ft. (stiffness),

$W_1 = W_2 = g$ lb. wt.,

$q = 1$ radian per sec.,

$F_1' = 1$ lb. per ft. per sec.,

$F_2' = 2$ lb. per ft. per sec.

(The above values are obviously absurd practically, but we are here concerned only with the interpretation of the solution.)

Then

$$\text{(i)} \quad -A + iA + (A-B) = P,$$

$$\text{(ii)} \quad -B + i \,.\, 2B + (B-A) + B = 0,$$

or

$$iA - B = P,$$

$$(1+2i)B - A = 0,$$

whence, solving for A and B,

$$B=\frac{P}{(i-3)},$$

$$A=\frac{P(1+2i)}{(i-3)}.$$

Now to obtain the magnitudes of the vectors represented by A and B, i.e. the amplitudes y_1 and y_2, together with their respective phase angles with the disturbing force, we must get these expressions in the form of $(a-ib)P$, and this can always be done by a little "juggling", as follows (remembering that $i^2=-1$):

$$A=\frac{P(1+2i)}{(i-3)}\times\frac{(i+3)}{(i+3)}=-\frac{P}{10}(7i+1)=\frac{P}{10}(-1-7i).$$

Thus the magnitude of the vector

$$y_1=\frac{P}{10}\sqrt{1+49}=\frac{P}{10}\sqrt{50},$$

and the phase angle with the disturbing force vector P has a tangent of 7/1.

Fig. 67 shows the result, and the directions of the components due to the negative sign are to be noted. Similarly,

$$B=\frac{P}{(i-3)}\times\frac{(i+3)}{(i+3)}=-\frac{P}{10}(3+i)=\frac{P}{10}(-3-i).$$

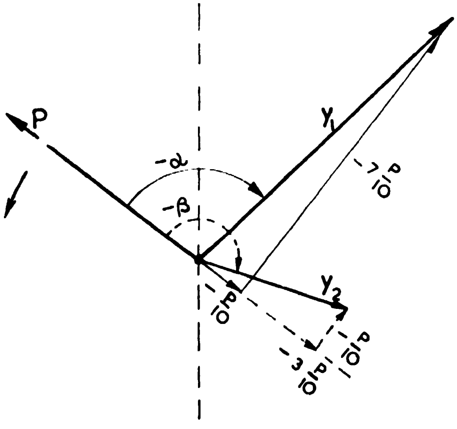


FIG. 67.

Thus the magnitude of the vector

$$y_2 = \frac{P}{10}\sqrt{9+1},$$

and the phase angle with vector P has a tangent of 1/3.

Every problem can be solved in this manner, although the process may become very involved where there are a large number of loads with different dampings, etc. Here we have considered each load subjected to a separate damping as indicated by the dashpots in the diagram, but we might have a frictional coupling between two loads with or without an elastic coupling, or again a frictional coupling between a load and its elastic coupling, etc. Two such systems are represented diagrammatically in Fig. 68*a* and *b*, and the frictional resistance will here be proportional to the relative velocity between the cylinder and the piston of the dashpots. Thus if x_1 = the displacement of the piston and x_2 the displacement of the cylinder, the frictional force will be $\pm F' \cdot \left(\frac{dx_1}{dt} - \frac{dx_2}{dt}\right)$ according as the datum is taken on the piston or the cylinder side of the connexion, and as a vector this expression may be represented by

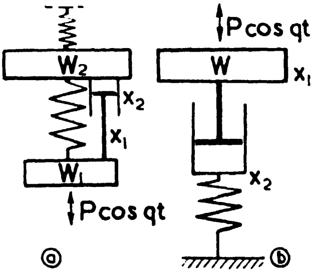


FIG. 68.

$$\pm iF'q(A-B)e^{iqt}.$$

A further example of this method is given in the theory of the friction damper on page 248.

Forced torsional oscillations of a shaft system. No damping. There should be no difficulty in transforming the equations developed for a linear system into the corresponding equations for torsional oscillation, and as the determination of the amplitudes in torsional oscillation are of supreme importance an example will be worked out, from which it will be seen that a tabular method similar to that on page 83 can conveniently be employed when the number of loads is large.

Let us consider a system comprising the first three loads and their retaining shafts of Fig. 30 and let us assume that the disturbing torque is $T \cos \sqrt{502} \,.\, t$ and acts on the second load.

[It will be remembered that $\sqrt{502}/2\pi$ was the one-node natural frequency of the four-load system of Fig. 30 for which the tabular solution was illustrated, but, as n^2 for our three-load system will not be this value, we have chosen it for q^2 because most of the calculations have already been made in the previous table.]

Let the motion of the loads be of the form $\theta = a \cos qt$, so that the equations of motion after changing over to amplitudes will be

$$\frac{I_1}{g} a_1 q^2 = T_{01}(a_1 - a_2),$$

$$\frac{I_2}{g} a_2 q^2 = T_{01}(a_2 - a_1) + T_{02}(a_2 - a_3) - T,$$

$$\frac{I_3}{g} a_3 q^2 = T_{02}(a_3 - a_2).$$

Note that, by the addition of these equations, the sum of the inertia torques is equal to $-T$.

We could solve for the amplitudes as before, but let us instead work along the shaft from load (1) in exactly the same manner as for natural oscillations except that we include the disturbing torque T when we come to it. The student should work through the table on p. 191 as before and note the differences now introduced.

$$\text{Inertia torque due to load (1)} = \frac{I_1}{g} q^2 a_1 = \underline{1 \cdot 83 \times 10^6 \,.\, a_1}.$$

$$\text{Twist in shaft (1)} = (a_1 - a_2) = \frac{1}{T_{01}} \left(\frac{I_1}{g} q^2 a_1 \right) = 0 \cdot 583 a_1 ;$$

$$\therefore \; a_2 = a_1 - \cdot 583 a_1 = 0 \cdot 417 a_1 ;$$

TABLE I

$q^2 = 502$ Torque $T \cos qt$ at load I_2

Load	$\frac{I}{g}$	$\frac{I}{g} \cdot q^2$	*Load Amplitude* a	$\frac{I}{g} \cdot q^2 a$	*Total Torque transmitted to next shaft*	T_0	$\frac{1}{T_0}$ [*Total Torque*]
1	3,650	$1{\cdot}83 \times 10^6$	a_1	$1{\cdot}83 \times 10^6 a_1$	$1{\cdot}83 \times 10^6 a_1$	$10^6\pi$	$\cdot 583a_1$
2	1,825	$\cdot 915 \times 10^6$	$\cdot 417a_1$	$\cdot 382 \times 10^6 a_1$	$2{\cdot}212 \times 10^6 a_1 + T$	$10^6\pi$	$\cdot 702a_1 + T/10^6\pi$
3	1,825	$\cdot 915 \times 10^6$	$-\cdot 285a_1 - T/10^6\pi$	$-\cdot 261 \times 10^6 a_1 - \cdot 2915T$	$1{\cdot}951 \times 10^6 a_1 + \cdot 7085T$		

TABLE II

$q^2 = 502$ Torque $T \cos qt$ at load I_1

Load	$\frac{I}{g}$	$\frac{I}{g} \cdot q^2$	*Load Amplitude* a	$\frac{I}{g} \cdot q^2 a$	*Total Torque transmitted to next shaft*	T_0	$\frac{1}{T_0}$ [*Total Torque*]
1	3,650	$1{\cdot}83 \times 10^6$	a_1	$1{\cdot}83 \times 10^6 a_1$	$1{\cdot}83 \times 10^6 a_1 + T$	$10^6\pi$	$\cdot 583a_1 + T/10^6\pi$
2	1,825	$\cdot 915 \times 10^6$	$\cdot 417a_1 - T/10^6\pi$	$\cdot 382 \times 10^6 a_1 - \cdot 2915T$	$2{\cdot}212 \times 10^6 a_1 + \cdot 7085T$	$10^6\pi$	$\cdot 702a_1 + \cdot 7085T/10^6\pi$
3	1,825	$\cdot 915 \times 10^6$	$-\cdot 285a_1 - 1{\cdot}7085T/10^6\pi$	$-\cdot 261 \times 10^6 a_1 - \cdot 496T$	$1{\cdot}951 \times 10^6 a_1 + \cdot 2125T$		

$$\therefore \text{ Inertia torque of load (2)} = \frac{I_2}{g} q^2 a_2 = 0{\cdot}915 \times 10^6 \times 0{\cdot}417 a_1$$
$$= \underline{0{\cdot}382 \times 10^6 . a_1.}$$

Then the total torque transmitted to shaft (2) $= \Sigma \frac{I}{g} q^2 a + T$

$$= [(1{\cdot}83 + 0{\cdot}382) \times 10^6 a_1] + T = 2{\cdot}212 \times 10^6 a_1 + T\,;$$

$$\therefore \text{ twist of shaft (2)} = (a_2 - a_3) = \frac{1}{T_{02}} \times \text{transmitted torque}$$
$$= \frac{1}{10^6 \pi} [2{\cdot}212 \times 10^6 a_1 + T]$$
$$= 0{\cdot}702 a_1 + \frac{T}{10^6 \pi}\,;$$

$\therefore$ amplitude of load (3) $= a_2 - (a_2 - a_3)$,

i.e. $$a_3 = -0{\cdot}285 a_1 - \frac{T}{10^6 \pi}\,;$$

$$\therefore \text{ inertia torque of load (3)} = \frac{I_3}{g} q^2 a_3 = 0{\cdot}915 \times 10^6 [a_3]$$
$$= \underline{-0{\cdot}261 \times 10^6 a_1 - 0{\cdot}2915 T,}$$

and, as this is the last load of the system, the total torque that is transmitted beyond this load is zero, i.e. $\Sigma \frac{I}{g} q^2 a + T = 0$, which is another way of saying that the sum of the inertia torques $= -T$ as noted above.

Thus

$$\Sigma \frac{I}{g} q^2 a = 1{\cdot}951 \times 10^6 a_1 - 0{\cdot}2915 T = -T,$$

or $$1{\cdot}951 \times 10^6 a_1 = -0{\cdot}7085 T,$$

so that $$a_1 = -0{\cdot}3638 \times 10^{-6} T \text{ radians}\,;$$

$$\therefore a_2 = -0{\cdot}1516 \times 10^{-6} T,$$
$$a_3 = -0{\cdot}2144 \times 10^{-6} T.$$

There are thus no nodes, and the displacements of the loads are here all 180 degrees out of phase with the disturbing torque.

The most important example in practice is that of a multi-cylinder I.C. engine in which the shaft system is subjected to simple harmonic component disturbing torques arising *from each cylinder*, and the resultant forced vibration would be obtained by the superposition of the motions calculated for a single disturbing torque applied at each cylinder in turn, but it should be pointed out that a given harmonic component of any one cylinder is not necessarily in phase with the same harmonic of any other cylinder, the phase depending on the order in which the cylinders come into action. This investigation is considered in detail in a later section.

Table II gives the calculation for the same harmonic torque applied at load (1), and the corresponding amplitude

$$a_1 = -0{\cdot}1089 \times 10^{-6} T.$$

EXAMPLES III

CHAPTERS VI AND VII

1. The mass of the reciprocating parts of a single cylinder engine is 50 lb. and the mass of the whole engine 1,000 lb. The engine rests on springs which are compressed 6 in. by its weight. Neglecting the obliquity of the connecting rod, determine the final amplitude of the vertical oscillations of the engine when running at 150 r.p.m. Stroke 1 ft. [Liv. U.]

The simple harmonic inertia reaction of the reciprocating parts is the disturbing force, its maximum value being (see page 216)

$$P=\frac{w}{g}\,\omega^2R=\frac{50}{32}\cdot\frac{4\pi^2\,.\,150^2}{60^2}\times\frac{1}{2}\text{ (ft. units)}.$$

Amplitude of forced vibration $=\dfrac{f}{n^2-q^2}$,

$$f=\frac{Pg}{W}=\frac{Pg}{1{,}000}=\frac{5\pi^2}{8},$$

$$n^2=\frac{Fg}{W}=\frac{g}{\text{static deflexion}}=\frac{32\times 12}{6}=64\,;$$

$$\frac{q}{2\pi}=\text{frequency of disturbing force}=\frac{150}{60}\,;$$

$$\therefore\ q^2=25\pi^2\,;$$

$$\therefore\ \text{amplitude}=-0{\cdot}0338\text{ ft.}$$

2. A mass of 32 lb., supported by an elastic structure, having a natural frequency of 10 per sec., is at rest in its mid position. A simple harmonic disturbing force of frequency 7·5 per sec. is applied, rising from zero to its maximum value of 100 lb. wt.

Determine the displacement of the load after the first $\frac{1}{100}$ sec. [Liv. U.]

Ans. 0·0007 ft.

The information shows that the disturbing force is $P\sin qt$,

whence $$x=\frac{f}{n^2-q^2}\left[\sin qt-\frac{q}{n}\sin nt\right].$$

3. A load of 320 lb., supported by an elastic structure, is free to perform vertical linear vibrations, having a natural frequency of $5/2\pi$ per sec. A simple harmonic vertical force of frequency $4/2\pi$ and maximum value 5 lb. is applied to the system, and at the instant of measuring time the load is situated 2 in. above, and is moving towards, its rest position, its velocity being 0·8 in. per sec. The force, at the same instant, has a magnitude of 4 lb. and is acting downwards and increasing. Determine the displacement after a lapse of 1 sec. [Liv. U.]

Ans. 1·192 in.

The information shows that force and displacement vectors are at angles e and α respectively with the projection line when $t=0$, i.e.

$$x = a\cos(nt+\alpha) + \frac{f}{n^2-q^2}\cos(qt+e)$$

$$= A\cos nt + B\sin nt + \frac{f}{n^2-q^2}\cos(qt+e).$$

$$\cos e = -\tfrac{4}{5}\,;\quad \therefore\ e = 180^\circ - 36^\circ\,52'\,;$$

$$a = \sqrt{A^2+B^2} = 2{\cdot}54\text{ in.}\,;\quad \tan\alpha = -\frac{B}{A} = 0{\cdot}0631\,;$$

$$\therefore\ \alpha = -3^\circ\,36'.$$

This shows that vector a is on the other side of the projection line, the body just reaching its maximum displacement upward due to the natural motion. The superimposed forced motion causes resultant motion to be downward.

4. If the body in problem (1) Examples II is subjected to a simple harmonic disturbing force of period $1\frac{1}{4}$ sec., what is the time lag between the application of the maximum value of the disturbing force and the body reaching its maximum amplitude? Also, if the resisting medium is removed, what will be the percentage alteration in amplitude of the forced vibration? *Ans.* 0·062 sec.; 5·04%.

5. A mass attached to a vertical spring vibrates in a medium whose resistance varies as the velocity. When the support is stationary the period of vibration is 1·5 sec., and the ratio of two consecutive maximum displacements on the same side of the mean is 5 : 1.

The support is now given a simple harmonic motion in a vertical plane of amplitude 1 in. and period 2 secs. Determine the amplitude of the forced vibration of the weight. [Liv. U.]

(*a*) *Support fixed.*

$$\text{Period} = \frac{2\pi}{\mu} = 1{\cdot}5\,;\quad \therefore\ \mu^2 = 17{\cdot}55.$$

$$\frac{x_1}{x_2} = 5 = e^{\frac{2k\pi}{\mu}}\,;$$

$$\therefore\ 2k = 2{\cdot}146.$$

Also $$\mu^2 = n^2 - k^2\,;\quad \therefore\ n^2 = 18{\cdot}693.$$

(*b*) *Support moving.*

$$A=\frac{n^2a}{n^2-q^2}\cos\alpha=\frac{n^2a}{\sqrt{(n^2-q^2)^2+4k^2q^2}}.$$

Now $$\tan\alpha=\frac{2kq}{n^2-q^2}=\cdot 7643 \quad (q^2=\pi^2);$$

$$\therefore\ \cos\alpha=0\cdot 7944;$$

$$\therefore\ \text{Amplitude } A=\frac{18\cdot 693\times 1\times 0\cdot 7944}{18\cdot 693-\pi^2}=\underline{1\cdot 68}\text{ in.}$$

6. A propeller of moment of inertia 32×10^6 lb. in.^{2} is free to vibrate on the end of a shaft 1 ft. in diameter and 57 ft. long, and may be considered as subjected to a simple harmonic disturbing torque of frequency 3 per sec. and of maximum value $0\cdot 5\times 10^6$ lb. in.

The water exerts a resistance varying with the velocity and this torque is $0\cdot 75\times 10^6$ lb. in. per unit velocity of 1 radian per sec.

Determine the amplitude of the resulting forced vibration. Take $C=5{,}000$ tons per sq. in.

If the damping medium were removed, by how much would the amplitude of the forced vibration be altered? [Liv. U.]

Ans. Amplitude $=0\cdot 035$ radians.
Damped amplitude = Undamped × $\cos\alpha$
($\cos\alpha=0\cdot 258$).

7. A mass of 8 lb. is supported by an elastic structure and causes a static extension of 3·84 in. This system is immersed in a medium which exerts a resistance of 3 lb. per unit velocity of 1 ft. sec., and is acted upon by a simple harmonic disturbing force of frequency 0·844 per sec. and maximum value 4 lb. Determine the amplitude of the forced vibration.

If the frequency of the disturbing force is altered so as to equal the natural frequency, show that the amplitude is *reduced* and explain why, on coincidence of the forced and natural frequencies, the amplitude is not a maximum. Show also that the previous value of the amplitude is the maximum.

$$n^2=\frac{Fg}{W}=\frac{g}{\text{static extension}}=\frac{32\times 12}{3\cdot 84}=100,$$

$$2k=\frac{F_1g}{W}=\frac{3\times 32}{8}=12,$$

$$f=\frac{Pg}{W}=\frac{4\times 32}{8}=16,$$

$$q=2\pi\times 0\cdot 844=5\cdot 3;$$

$$\therefore\ q^2=28\cdot 09;$$

$$\therefore \ \tan\alpha = \frac{2kq}{n^2 - q^2} = \frac{12 \times 5{\cdot}3}{100 - 28{\cdot}09} = 0{\cdot}882\ ;$$

$$\therefore \ \cos\alpha = 0{\cdot}7499\ ;$$

$$\therefore \ \text{amplitude} = \frac{f}{n^2 - q^2}\cos\alpha = \frac{16}{100 - 28{\cdot}09} \times 0{\cdot}7499$$

$$= \underline{0{\cdot}1667 \text{ ft.}}\ ;$$

alternatively,

$$\text{amplitude} = \frac{f}{2kq}\sin\alpha, \quad \text{or} \quad \frac{f}{\sqrt{(n^2 - q^2)^2 + 4k^2q^2}}.$$

When $q = n,\ \alpha = 90°\ ;\ \therefore\ \sin\alpha = 1\ ;$

$$\therefore \ \text{amplitude} = \frac{16}{12 \times 10} = \underline{0{\cdot}1333 \text{ ft.}}$$

(See curves, page 160.)

Provided $\frac{2k}{n}$ is less than $\sqrt{2}$

$$\text{maximum amplitude} = \frac{f}{2k\mu},$$

where

$$\mu^2 = n^2 - k^2$$

$$= 100 - 36 = 64\ ;$$

$$\therefore \ \text{max. amplitude} = \frac{16}{12 \times 8} = \underline{0{\cdot}1667 \text{ ft.}}$$

8. A single load of weight 2 lb. when attached to an elastic structure causes a static deflexion of ½ in. The system is immersed in a medium, the resistance of which varies as the velocity, and when a simple harmonic disturbing force is applied to the load it is observed that the maximum amplitude of the forced vibration occurs when the frequency of the disturbing force is half the natural frequency of the system undamped. Determine the resistance per unit velocity. [Liv. U.]

Ans. 2·12 lb. per ft. per sec.

9. A mass of 10 lb. when attached to a vertical spring extends it 1 in. The system is now immersed in a medium whose resistance varies as the velocity, and the support of the spring is given a simple harmonic vertical oscillation of amplitude 2 in. and period equal to the natural period of the spring undamped, and the amplitude of the resulting vibration of the load is found to be 2 in.

Show that, if the support is now fixed, the system can undergo a damped vibration, and determine the period of this vibration and the ratio of two consecutive maximum displacements on opposite sides of the mean. Show also that the maximum amplitude occurs when the frequency ratio is $1/\sqrt{2}$ and determine its value. [Liv. U.]

Ans. 0·37 secs.; 6 : 1; 2·3 in.

✓ **10.** The static deflexion of a spring produced by a load of 160 lb. was 1·2 in. When a simple harmonic disturbing force, of maximum value 40 lb. and frequency equal to the natural frequency of the system undamped, was applied to the load when in a medium whose resistance varied as the velocity, the amplitude was found to be the same as that which would be produced under static application of the maximum disturbing force.

Determine (*a*) this amplitude; (*b*) the frictional resistance per unit velocity; (*c*) the maximum amplitude possible; and (*d*) the frequency ratio at which it would occur. [Liv. U.]

Ans. (*a*) 0·3 in.;
(*b*) $F_1 = 89{\cdot}25$ lb. per ft. per sec.;
(*c*) 0·347 in.; (*d*) $1/\sqrt{2}$.

✓ **11.** A machine for producing vibrations consists of an electric motor mounted on springs. To the motor shaft is attached a disk which is put out of balance by a mass of 1 lb. attached at a radius of 6 in. Vibration is possible in a vertical plane only and a dashpot is introduced to limit the amplitude, friction being proportional to the velocity. When the motor is run at 60 r.p.m., it is observed that the out-of-balance mass is passing through its highest or lowest position relative to the motor shaft, while the motor is passing through its mean position in one direction or the other, and the amplitude of the vibration is 1 in. If the total mass of the vibrating load is 32 lb., determine:

(*a*) the natural frequency of the system undamped;
(*b*) the value of the damping coefficient;
(*c*) the amplitude of the vibration with the same damping at 50 r.p.m.
(*d*) the amplitude at this speed if damping were removed.

[Liv. U.]

Ans. (*a*) 1 per sec.;
(*b*) $F_1 = 3\pi/8$ lb. per ft. per sec.;
(*c*) 0·38 in.; (*d*) 0·428 in.

Note that the information given states that the disturbing force and displacement vectors are at 90°.

12. A single load supported by an elastic structure is subjected to a simple harmonic force of frequency $10/2\pi$ per sec., whilst in a medium whose resistance is 5 lb. per in. per sec. If the energy supplied to overcome damping is 50 ft. lb., what will be the resulting amplitude of the load? [Liv. U.]

Ans. 1·95 in.

13. A single cylinder vertical petrol engine of total weight 75 lb. is mounted upon a steel chassis frame and causes a static deflexion of 0·12 in. The reciprocating parts weigh 1·6 lb., have a stroke of 3 in., and may be considered to move with simple harmonic motion

When running under resonant conditions it was found that the amplitude of vibration of the engine on its mounting was 0·1 in.

Assuming that damping resistance varies as the velocity, determine the amplitude of the forced motion when the engine runs at half this resonant speed. [Liv. U.]

Ans. 0·0158 in.

14. Fig. (*a*) represents a disk *I*, of moment of inertia 32 lb.-ft.^{2} units, keyed to the end of a shaft of torsional stiffness $T_0 = 1{,}600$ lb. ft. per radian which is attached to the frictional damper *A*.

If this damper is given a simple harmonic angular motion of frequency equal to the natural frequency of the loaded shaft, determine (*a*) the damping coefficient required to limit the amplitude of the load to twice that of the damper, friction varying as the velocity; and (*b*) the relative amplitude of load to support. [Liv. U.]

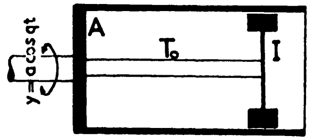


FIG. III (*a*).

Ans. (*a*) 23 lb. ft. per radian per sec.
(*b*) $\sqrt{3} \times$ amplitude of damper.

Motion of support $y = a \cos qt$.

$$\frac{I}{g}\frac{d^2\theta}{dt^2} + T_1\frac{d\theta}{dt} + T_0\theta = T_0 y + T_1\frac{dy}{dt}$$

$$= \sqrt{(T_0 a)^2 + (T_1 aq)^2} \,.\, \cos(qt + e) = P\cos(qt + e).$$

When $q = n, \quad P = T_1 qA$.

15. A load of 100 lb. when attached to the free end of a vertical spring, which is fixed at the other end, causes a static deflexion of 0·64 in.

When the support is given a S.H.M. of circular frequency 20 radians per sec., the amplitude of the load is found to be 6 in., damping being negligible; but when damping varying as the velocity of the load is introduced, the amplitude is found to reduce to 3 in.

Determine:

(*a*) the amplitude of the support;
(*b*) the damping coefficient;
(*c*) the phase angle between the vectors which generate the motions of the load and the support when damping is in operation;
(*d*) the relative amplitude between the load and the support;
(*e*) the phase angle between the relative amplitude vector and the vector generating the motion of the load;
(*f*) the value of the magnification factor with damping.

Ans. (*a*) 2 in.; (*b*) 54·1 lb. per ft. per sec.;
(*c*) 60°; (*d*) 4·36 in.
(*e*) $\sin^{-1} 0{\cdot}397$; (*f*) 1·5.

16. A load of 64 lb. when attached to an elastic support causes a static extension of 1 in. When a S.H. disturbing force of maximum value 53 lb. and frequency equal to the natural frequency acts on the load, while in a medium whose resistance varies as the velocity, the amplitude is found to be 0·88 in. Determine:

(*a*) the frictional resistance per unit velocity;

(*b*) the frequency of the disturbing force which would cause the amplitude *without* damping to be 1 in.;

(*c*) the amplitude which would occur with this frequency of the disturbing force if the damping medium was restored;

(*d*) the amplitude of the S.H.M. which, if given to the support, would produce the same vibration of the load as was given by this disturbing force. [Liv. U.]

Ans. (*a*) $F_1 = 36{\cdot}8$ lb. per ft. per sec.;
(*b*) 1·3 per sec.;
(*c*) 0·902 in.; (*d*) 0·83 in.

17. Two loads W_1, W_2 are arranged to vibrate in a vertical plane, being connected by a spring of stiffness F_1, while W_2 is attached to the support by a spring of stiffness F_2.

When the load W_2 was subjected to a simple harmonic disturbing force of frequency $6/2\pi$ vibrations per sec., it was found to remain stationary while the amplitude of W_1 was 1 in.

The frequency of this disturbing force was then changed, when the amplitudes of W_1 and W_2 were found to be 1 in. and 2 in. respectively and a node was formed in the connecting spring, the motion of W_1 being in phase with the disturbing force.

Determine (*a*) this frequency, (*b*) the maximum value of the disturbing force, and (*c*) the stiffnesses of the springs. [Liv. U.]

[$W_1 = 32$ lb. $W_2 = 64$ lb.]

Ans. (*a*) $6\sqrt{3}/2\pi$; (*b*) $P = 3$ lb.
(*c*) $F_1 = 36$ lb. per ft.; $F_2 = 3{\cdot}36$ lb. per ft.

18. The rotating and reciprocating parts of a four-stroke engine may be considered as equivalent to a single flywheel of moment of inertia 1,600 lb. ft.^{2} The engine runs at 600 r.p.m. and drives a rotating system of moment of inertia 800 lb. ft.^{2} by means of a shaft of torsional stiffness $20{,}000\pi^2$ lb. ft. per radian. The fundamental harmonic of the engine torque is

$$T = 50\pi^2 \cos(qt + e) \text{ lb. ft.}$$

Determine the amplitudes (degrees) of vibration of the driving and driven flywheels due to this disturbing torque. [Liv. U.]

Ans. $a_1 = -0{\cdot}364$ deg.
$a_2 = -0{\cdot}415$ deg.

Fundamental harmonic ½ cycle per revolution of engine;

$$\therefore \frac{q}{2\pi} = 300 \text{ per minute.}$$

19. Fig. (*b*) represents a light cage W_1 attached by a spring of stiffness F_1 to a roller which can travel over a sinusoidal track. Mounted inside the cage, on springs of total stiffness F_2, is a recording drum W_2, which will record the relative movement between it and the cage by means of the pointer A.

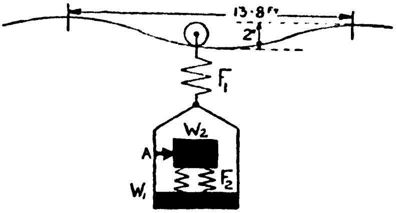


FIG. III (*b*).

If the distance between consecutive crests of the track is 13·8 ft., and the height between crest and hollow is 2 in., determine :

(*a*) the relative amplitude when speed along track is 15 miles per hour, and

(*b*) the track speed which will make the amplitude of the cage W_1 zero, and the corresponding amplitude of the drum W_2.

$W_1 = 2W_2 = 8$ lb. $F_1 = 2F_2 = 10$ lb. per in.

[Liv. U.]

15 m.p.h. = 22 ft. per sec.

$$\text{Period of support} = \frac{2\pi}{q} = \frac{13\cdot 8}{22}; \quad \therefore\ q = 10 \quad \text{or} \quad q^2 = 100.$$

Motion of support $y = a \cos qt \quad (a = 1 \text{ in.})$

$$\frac{W_1}{g} q^2 y_1 = F_1 y_1 - F_1 a + F_2(y_1 - y_2);$$

$$\frac{W_2}{g} q^2 y_2 = F_2(y_2 - y_1);$$

whence $y_1 = 0\cdot 462$ in., $y_2 = 0\cdot 583$ in.; $\therefore\ y_1 - y_2 = 0\cdot 121$ in.

$$\text{If } y_1 = 0,\ q^2 = \frac{F_2 g}{W_2} = 480; \quad \therefore\ \frac{2\pi}{q} = 0\cdot 287 \text{ sec.,}$$

or speed = 32·8 m.p.h.

$$F_1 a = -F_2 y_2; \quad \therefore\ y_2 = -2 \text{ in.}$$

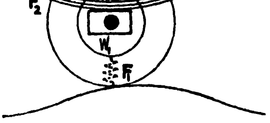


FIG. III (*c*).

20. A trailer has two rubber-tyred wheels, Fig. (*c*). The total load ($W_1 + W_2$) on the two tyres is 2,560 lb., while the portion W_2 on the spring system is 1,600 lb. The stiffness of the two tyres combined is 30,000 lb. per ft., and that of the spring system 9,000 lb. per ft. The trailer is drawn over a sinusoidal track, the height between hollow and crest being 2 in., and the distance between crests 31·42 ft.

Determine (*a*) the resonant speeds of the trailer; and (*b*) the ratio of the amplitudes of the body and axle when a wave is covered in $\pi/2$ secs. [Liv. U.]

Ans. (*a*) 58; 183·5 ft./sec.; (*b*) $y_1/y_2 = 0{\cdot}91$.

21. A motor-car has a body weighing 2,400 lb., which compresses the spring system, consisting of four equal springs at equal distances from the C.G., through 4 in. Four shock absorbers are fitted between the body and the chassis, and each exerts a resistance of 10 lb. per in. per sec. If the car is running over a sine-wave track of 1 in. amplitude and 44 ft. between crests, determine the amplitude of the body from its rest position for a speed of 30 m.p.h. [Liv. U.]

Ans. 1·5 in.

22. A mass of 60 lb., when attached to a light spring system, causes a static deflexion of 6 in. Vibration is possible in a vertical plane only and, to limit the amplitude, dashpots are fitted between the load and the spring support, the frictional resistance exerted being 1 lb. wt. per in. per sec.

If the support is given a S.H.M. of amplitude 1 in. and frequency $6/2\pi$, determine (1) the amplitude of the load; (2) the relative amplitude between load and support; (3) if frequency of disturbing force is made equal to the natural frequency, what would be the amplitude of the load?; (4) if, in the above, the dashpot has been fixed between the load and the ground, what would have been the amplitude of the load, when $q = 6$, $q = n$?

$$n^2 = \frac{32 \times 12}{6} = 64, \quad \delta_{st} = \frac{W}{F}; \quad \therefore \; F = \frac{60}{6} = 10 \text{ lb. per in.}$$

$F_1 = 1$ lb. per in. per sec.

$q = 6$. $\qquad P = \sqrt{(Fa)^2 + (F_1 qa)^2} = \sqrt{136}$.

$a = 1$ in. $\qquad$ Also from vectors for L.H.S.

$$P^2 = \left(FA - \frac{W}{g}\,q^2 A\right)^2 + (F_1 A q)^2$$

$$\therefore \; \underline{A = 1{\cdot}57 \text{ in.}}$$

(2) $\tan\alpha = \dfrac{\dfrac{F_1 g}{W} \cdot q}{n^2 - q^2}; \quad \therefore \; \alpha = 53^\circ\, 55' \quad \Big| \quad \therefore \; (\alpha - e) = 23^\circ;$

$\tan e = \dfrac{F_1 q}{F}; \quad \therefore \; e = 30^\circ\, 58' \quad \Big| \quad \therefore \; \cos(\alpha - e) = 0{\cdot}920;$

$$R^2 = A^2 + a^2 - 2aA\cos(\alpha - e);$$

$$\therefore \; R = 0{\cdot}76 \text{ in.}$$

(3) $q=n.\qquad P=F_1Aq=\sqrt{(Fa)^2+(F_1qa)^2}=\sqrt{100+64}$;

$$\therefore\ \underline{A=1\cdot 6\text{ in.}}$$

(4) Dashpot between load and ground.

(*a*) $q=6.\qquad P=Fa=10$ lb.

$$\therefore\ A^2=\frac{100}{55\cdot 15}=1\cdot 8,\quad \text{i.e. } \underline{A=1\cdot 34\text{ in.}}$$

(*b*) $q=n\qquad P=Fa=F_1Aq,$

$$\text{i.e. } 10=A\times 8\quad\text{or}\quad \underline{A=1\cdot 25\text{ in.}}$$

23. A two-load system elastically mounted is subjected to a periodic disturbing force $P\cos qt$. In the vector solution by complex quantities, the vector counterparts of the amplitudes y_1, y_2 of the loads are found to be

$$A=P\cdot\frac{(2+3i)}{(i-1)},$$

$$B=P\cdot\frac{(3+2i)}{(i-1)},\ \text{respectively.}$$

Determine the magnitudes and phase positions of these vectors relative to the disturbing force vector as datum.

$$\textit{Ans.}\quad y_1=y_2=\frac{P}{2}\cdot\sqrt{26}.$$

$$\alpha=78\cdot 6^\circ\text{ lag};\quad \beta=(180^\circ-78\cdot 6^\circ)\text{ lag.}$$

24. Two masses of weights W_1, W_2 lb. are coupled by an elastic connexion of stiffness F_2, and damping equivalent to a dashpot is also in existence between them (as in Fig. 68*a*), the resistance being F_1' per ft. per sec. W_2 is coupled to the support by a spring of stiffness F_1, and W_1 is coupled to the ground support through a dashpot (of resistance F_2' per unit velocity) which is in series with a spring of stiffness F_3 (as in Fig. 68*b*).

If a disturbing force $P\cos qt$ acts on W_2, determine the amplitudes y_1, y_2, y_3 and the phase angles α, β, γ of the loads W_1, W_2 and the dashpot F_2', given

$W_2=32$ lb.

$W_1=64$ lb.

$q=10$ radians per sec.

$F_1=100$ lb. per ft.

$F_2=200$ lb. per ft.

$F_3=400$ lb. per ft.

$F_1'=10$ lb. per ft. per sec.

$F_2'=20$ lb. per ft. per sec.

$P=100$ lb. wt.

Ans. If
$$\begin{aligned} x_1 &= y_1 e^{i(qt-\alpha)} = Ae^{iqt}, \\ x_2 &= y_2 e^{i(qt-\beta)} = Be^{iqt}, \\ x_3 &= y_3 e^{i(qt-\gamma)} = Ce^{iqt}, \end{aligned}$$

then
$$\begin{array}{lll} A = \dfrac{(1-6i)}{10} & \text{and} & y_1 = \sqrt{37}/10 \text{ ft.} \\ B = \dfrac{(-3-4i)}{10} & & y_2 = \sqrt{25}/10 \text{ ft.} \\ C = \dfrac{(1-2i)}{10} & & y_3 = \sqrt{5}/10 \text{ ft.} \end{array}$$

25. Determine the magnitudes and vector phase positions of the amplitudes of the two-load system shown in Fig. 66, page 186, if the dashpot of frictional resistance F_2' per unit velocity is coupled ***between the two loads***, the other dashpot remaining as shown, given

$W_1 = 32$ lb. $\quad F_1' = 20$ lb. per ft. per sec.

$W_2 = 64$ lb. $\quad F_2' = 10$ lb. per ft. per sec.

$F_1 = 200$ lb. per ft. $\quad P = 100$ lb.

$F_2 = 100$ lb. per ft. $\quad q = 10.$

Ans. $y_1 = \dfrac{\sqrt{5}}{5}$ ft.; $y_2 = \dfrac{\sqrt{2}}{5}$.

CHAPTER VIII

SOME PRACTICAL APPLICATIONS OF VIBRATIONS

(*a*) **Engine indicator** (pencil type). This is a loaded spring system which is acted upon by periodic pressure fluctuations. In order that the piston movement should follow accurately these fluctuations the Magnification Factor must be unity, i.e. the engine speed (frequency of disturbing force) must be *low* compared with the natural frequency. This is one of the reasons why such an indicator is unsuitable for a high-speed engine.

(*b*) **Earthquake and other vibration recorders.** Fig. 69 (*a*) shows an arrangement for recording vertical vibrations. If the *natural* frequency of the load W on its spring is *low* compared with the frequency of the vibrations transmitted to the support, the load is stationary in space and acts as a fixed datum for recording the movement of everything about it.

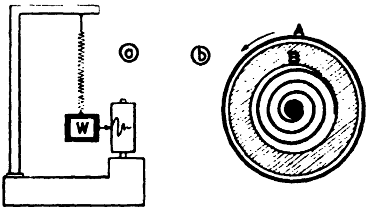


FIG. 69.

A difficulty arises in practice in that to get a low natural frequency compared with the disturbance $W/F=\delta_{st}$, the static deflexion, must be large, e.g.:

$$\text{frequency}=\frac{1}{2\pi}\sqrt{\frac{Fg}{W}}=\frac{1}{2\pi}\sqrt{\frac{g}{\delta_{st}}},$$

and for a frequency of (say) 1 per sec., $\delta_{st}=\frac{32}{4\pi^2}=0\cdot8$ ft.,

whilst for $\frac{1}{4}$ per sec., $\delta_{st}=0\cdot8\times16=13$ ft.!

and the spring system necessary to be flexible enough and yet be strong enough to withstand the heavy load is, in general, too bulky to be practical. This aspect will be referred to again when discussing flexible connexions for reducing the transmission of vibrations in the next chapter. We may overcome this difficulty in the case now under consideration, by mounting the system as shown in Fig. (*c*), Example 10, on page 108, so that the load W has *angular* motion about O.

The equivalent load at the spring to give the same *moment of inertia* about O is $W(l_1/l_2)^2$, i.e.

$$\text{frequency} = \frac{1}{2\pi}\sqrt{\frac{Fg}{W(l_1/l_2)^2}}.$$

With the same load W, for the same natural frequency, the spring need only be $(l_2/l_1)^2$ of its former length.

Now, at any given frequency $q/2\pi$, the record produced will be that of the motion of the load *relative to the support.*

With no damping the relative amplitude was shown to be

$$(A - a) = \frac{q^2}{n^2 - q^2} \cdot a \quad \text{(see page 170)}$$

and the ratio

$$\frac{\text{relative amplitude}}{\text{amplitude of support}} = \frac{q^2}{n^2 - q^2},$$

is called the *Response* of the instrument, since it shows how truly the record represents the motion of the support, which is that to be measured.

$$\left.\begin{array}{r} \text{When } q \text{ is small, Response} \doteqdot 0 \\ \text{Magnification Factor} \doteqdot 1 \end{array}\right\} \text{i.e. the load and recording drum move as a solid body.}$$

$$\left.\begin{array}{r} \text{When } q \text{ is large, Response} \doteqdot 1 \\ \text{M.F.} \doteqdot 0 \end{array}\right\} \text{i.e. the load is approximately stationary in space.}$$

With damping, we have seen that there is a phase difference between the vectors generating the motion of the load and the support, and the relative amplitude is their vector difference.

However, when q is large compared with n, this phase angle approaches 180 degrees, and the response is approximately unity as for no damping, so that the instrument will give an accurate record.

Upon the same principle depends an instrument—the Geiger Torsiograph—for measuring torsional oscillations (Fig. 69*b*). This consists essentially of a light pulley A, driven by a non-elastic belt from the point on the shaft at which the angular oscillations are to be measured. Within this pulley is a heavy flywheel B, free to turn on the same axis and coupled to A's spindle by a flexible flat spiral spring.

The natural frequency of the flywheel B on its springs is made as small as possible compared with the oscillations which will be transmitted to the " support " A, so that the flywheel will tend to rotate with *constant angular velocity* (cf. stationary in space), the oscillations of A not appreciably affecting the motion. The relative movement between the pulley and the flywheel is recorded by a pen on a strip of paper unwound from a drum by clockwork. On this strip is also recorded a time scale, a dot occurring at each revolution of the engine.

A series of records taken at different speeds from the shaft of a 6-cylinder diesel engine for a submarine are shown in Fig. 70*a*. A critical resonant speed had been calculated to occur at 500 r.p.m. and it will be seen that at 480 r.p.m. the vibrations were very pronounced, thus confirming the mathematical analysis. The slow wave upon which the shaft vibrations are superimposed is due to the pen-recording system which has a natural frequency of its own.

Fig. 70*b* shows some records taken from a steam turbine drive for a large passenger vessel and the absence of vibration is very marked, and although to rather a small scale, the student will probably be able to see the four main impulses per revolution due to a four-blade propeller.

As will be explained when dealing with engine vibration in a later chapter, each harmonic of the cylinder torque fluctuations can force a vibration which may be serious under resonant conditions. With a little practice the operator can tell by inspection of the record which harmonics are causing trouble, since he will soon become familiar with the general shapes of the resultant curves formed when two (or more) harmonics are superimposed.

For instance, the record for 440 r.p.m. is very similar to figure 5 (page 25) since it shows a marked beat, the amplitude varying from zero to a maximum. This will be the resultant, therefore, of two motions of nearly the same frequencies and equal amplitudes.

If these frequencies are $\frac{q_1}{2\pi}$, $\frac{q_2}{2\pi}$, it will be remembered that the frequency of the resultant approximate simple harmonic motion is $\frac{1}{2}(q_1+q_2)/2\pi$, and the beat frequency $(q_1-q_2)/2\pi$.

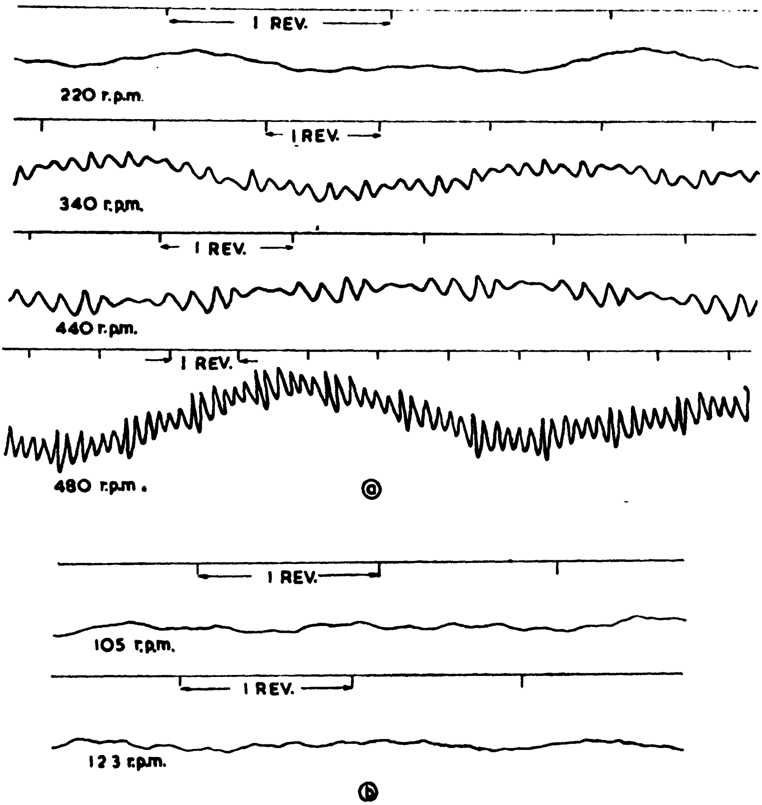


FIG. 70.

From the revolution calibration it will be seen that the beat period is in 1 revolution, i.e. a frequency of 440 per min., and there are 6 cycles of the resultant per revolution or 6 × 440 per min.

Thus

$$\frac{1}{2}(q_1+q_2)/2\pi = 6 \times 440,$$

$$(q_1-q_2)/2\pi = 440\,;$$

$$\therefore\ q_1+q_2 = 12 \times 440 \times 2\pi,$$

$$q_1-q_2 = 1 \times 440 \times 2\pi,$$

from which $$q_1 = 6\tfrac{1}{2} \times 440 \times 2\pi,$$

$$q_2 = 5\tfrac{1}{2} \times 440 \times 2\pi,$$

i.e. $$\frac{q_1}{2\pi} = 6\tfrac{1}{2} \times 440 \text{ cycles per min.}$$

$$\frac{q_2}{2\pi} = 5\tfrac{1}{2} \times 440 \text{ cycles per min.}$$

or the harmonics concerned have orders $6\frac{1}{2}$, $5\frac{1}{2}$ (cycles per revolution) respectively.

Thus it would appear that the 11th and 13th harmonics are predominant at this speed forcing vibrations of the same amplitude. More information, however, is really necessary and the matter is further discussed in the next chapter.

Fatigue testing machine. Fig. 71*a* shows the essentials of a machine which enables a constant periodic disturbance, independent of the inertia of linkages, etc., to be applied to a specimen.

A disk A carries an out-of-balance mass w and is driven by a belt from an electric motor B.

The disk is situated at the free end of a cantilever spring C of stiffness F, and vibrations can be transmitted to the specimen via the links D, E, causing alternating bending stresses.

The system is equivalent to Fig. 71*b*, where W is the equivalent load at the end of the cantilever which includes the inertia of the linkages D, E, and F_s is the stiffness of the specimen.

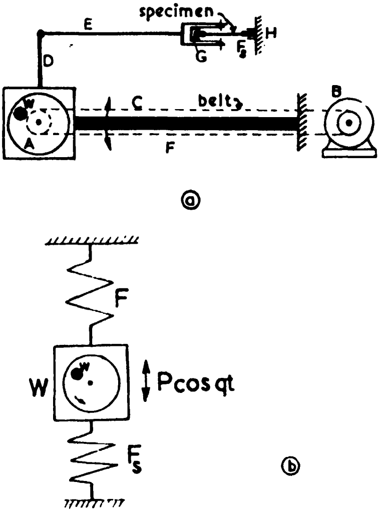


FIG. 71.

The out-of-balance disk is run at a speed equal to the natural frequency of W on its cantilever spring of stiffness F, i.e. $q^2 = \dfrac{Fg}{W}$.

We have

$$\frac{W}{g} d^2x/dt^2 = -(F+F_s)x + P\cos qt, \text{ where } P = \frac{w}{g} q^2 r;$$

$$\therefore \frac{W}{g} d^2x/dt^2 + (F+F_s)x = P\cos qt;$$

$$\therefore A = \frac{Pg/W}{(F+F_s)\frac{g}{W} - q^2} = \frac{Pg/W}{F_s g/W} = \frac{P}{F_s}.$$

Thus as F_s alters due to fatigue, the amplitude A varies to suit, the disturbing force remaining the same.

The arrangement as sketched shows the specimen subjected to pure bending, but in the actual machine twisting can also be introduced by rotating the head *H* about a vertical axis, the grip *G* being free to turn on the link *E* about a vertical axis passing through the centre of the specimen.

For a full description of the research for which the machine was designed the reader is referred to the paper " The Strength of Materials under Combined Alternating Stress " by H. J. Gough, D.Sc., F.R.S., M.I.Mech.E., and H. V. Pollard, A.M.I.Mech.E.*

Balancing machines for rotors. On p. 233 it is shown that the balance of any rotating system can be effected by the addition of two revolving masses in any two different parallel planes of revolution, or, in other words, the unbalance can be reduced to that of two forces in any two planes of revolution.

In Fig. 72*a*, which is a plan view, *A* is a small rotor mounted on ball bearings in a frame *B* which is capable of vibration about the horizontal knife edges *c,c*, and controlled by the vertical springs *D*. The position of the knife edges along the frame can be altered as required.

The most convenient planes in which to place the balancing masses will be at the ends of the rotor *E, F*, i.e. the unbalance is being reduced to two forces in these planes.

The method of operation is then as follows :

Place the knife edges *c,c* in the same plane as *E* (as shown) ; this will eliminate the effect of the unbalanced force in this plane, and any vibrations will be due to the unbalanced force in the other plane *F*.

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* *Proceedings of the Institution of Mechanical Engineers*, Vol. 131, 1395.

Run up the rotor above the resonant speed of the system on the springs *D* (by means of a friction drive from an electric motor), and allow the rotor to run down through this speed. Any unbalance in plane *F* will cause a violent vibration as it passes these conditions and the amplitude can be measured,

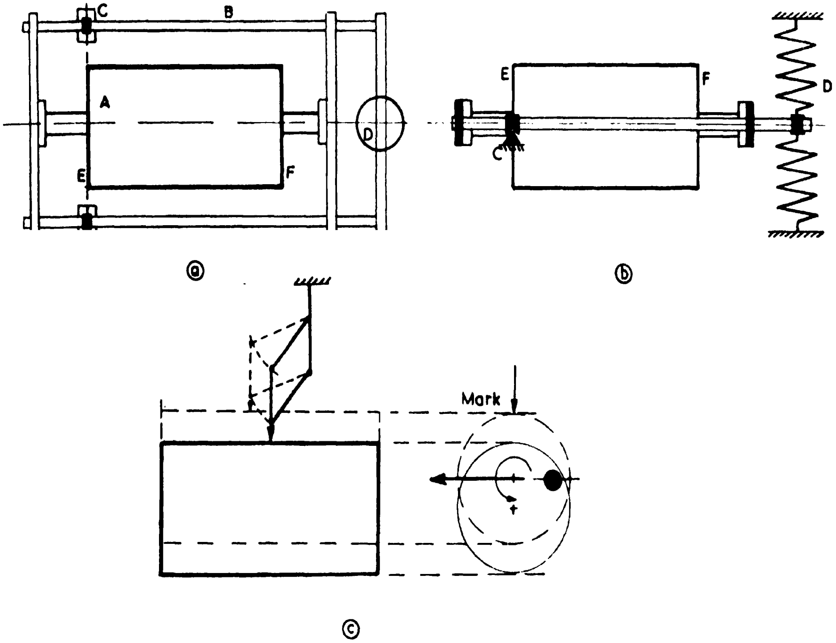

Fig. 72.

usually by an optical indicator. A small mass is now attached to the rotor in plane *F* (sometimes in holes provided and sometimes just a lump of putty stuck on), and runs are made with this mass in different circumferential positions until a position is found in which the amplitude is a minimum. By plotting the amplitude against the position, the position required can be found with the minimum of runs.

A further series of runs is now made but changing the magnitude of the mass in this position until (again by plotting) the magnitude to give zero amplitude is obtained.

The whole operation is then repeated with the knife edges in plane F to determine the balancing mass required in plane E.

Finally material is *removed* diametrically opposite these masses instead of making attachments.

The time taken to determine the position of a mass can be considerably shortened by the use of the scriber shown in Fig. 72*c*. This is pressed against a portion of the shaft or rotor which has been blackened and moves along the shaft describing portions of a spiral as the displacement increases. The scriber, after displacement, does not return, so that the end of the curve so drawn will represent the position of the shaft when at its maximum amplitude, i.e. at resonance. But under these conditions we know that the disturbing force vector is 90 degrees ahead of the displacement so that the balancing mass will be situated 90 degrees behind the end of the line scribed.

As the phase angle changes so rapidly on passing through resonant conditions, it is preferable to make two runs in opposite directions and bisect the angle between the two ends of the lines scribed, which will probably not be quite 180 degrees apart.

It should be pointed out that we are here considering the resonant conditions of the rotor system on the springs D, the shaft being assumed to be rigid, and not the " whirling " conditions discussed in Chapter XIII, which concerns the transverse vibrations of the rotor on the *shaft*, as elastic support. The stiffness of the springs D is so chosen that the resonant speed is low compared with the frequency of transverse vibration of the rotor on the shaft and therefore there is no danger of approaching the whirling conditions.

Packing machines. Many ingenious devices have been designed for deliberately introducing vibrations to assist the packing of sugar or other loose products, and an interesting paper on this subject—" The Use of Vibration as a Means of Industrial Drive " by C. A. M. Thornton, M.A., A.M.I.Mech.E. —is to be found in the *Proceedings* of the Institution of Mechanical Engineers.*

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* 1947, Vol. 157, War Emergency Issue, No. 25.

CHAPTER IX

FORCED VIBRATIONS OF A MULTI-CYLINDER ENGINE

HAVING discussed the general effects of a periodic disturbance on an elastic system, we shall now proceed to review one of the most important cases of potential synchronous oscillation, namely, a multi-cylinder engine.

A reciprocating engine can be considered as comprising two main elastic systems : (*a*) the engine itself bolted to its foundations, which may themselves be elastic and capable of being set into severe vibration, e.g. the hull of a ship ; and (*b*) the crankshaft and its attached masses.

Both systems may suffer serious vibrations due chiefly to the disturbing effect of the periodic torque fluctuations, which in an internal combustion engine are more pronounced than when steam is the working substance. Thus with the ever-increasing use of compression-ignition engines for marine and other purposes, the problem of the elimination of vibration is of supreme importance.

Consider a single cylinder engine—the moving parts consist of (1) revolving masses and (2) reciprocating masses.

The connecting rod has both revolving and reciprocating motion, and it is usual to replace the rod by an " equivalent dynamical system " of two masses in any calculations. These two masses would then cause exactly the same reactions as the rod itself, and the necesssary requirements are :

(1) the sum of the masses must equal the mass of the rod ;

(2) the centre of gravity of the masses must coincide with the C.G. of the actual rod.

These two conditions ensure the same linear accelerations of the C.G.'s at any instant, and to obtain the same angular acceleration about the C.G. we require that

(3) the moment of inertia of the masses about an axis through their C.G. and perpendicular to the plane of motion must equal the moment of inertia of the rod about a similar axis.

Actually an approximation is resorted to, which satisfies the first two conditions but leaves the third unfulfilled, the two masses being assumed concentrated at the gudgeon pin and crankpin respectively, the first therefore having purely reciprocating motion and the second purely revolving motion, their magnitudes being equal to the reactions at the respective pins when the rod is supported horizontally as indicated in Fig. 73. These two masses can thus be included in the reciprocating and revolving systems respectively.

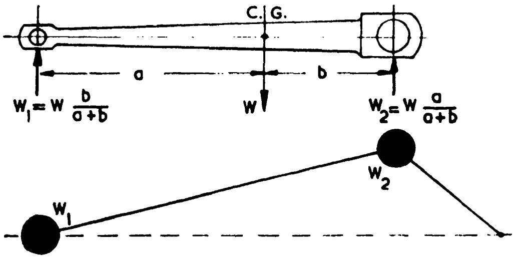


Fig 73..

(1) **Revolving masses.** These will set up centrifugal reactions, which, resolved horizontally and vertically, will cause simple harmonic disturbing forces in these planes of engine speed frequency.

(2) **Reciprocating masses.** These, due to their acceleration, will cause a corresponding reaction on the engine frame *in the line of stroke.*

To determine the force required to accelerate the reciprocating parts in terms of the angle turned through by the crank from the inner dead centre, Fig. 74.

Let R = the crank radius,

L = the length of the connecting rod,

x = the displacement of the piston from the inner dead centre.

With centre B and radius L, strike an arc AE.

Draw AD perpendicular to the line of stroke.

Then

$$\begin{aligned} x &= CE \\ &= CD + DE \\ &= (OC - OD) + DE \\ &= R(1 - \cos\theta) + DE. \end{aligned}$$

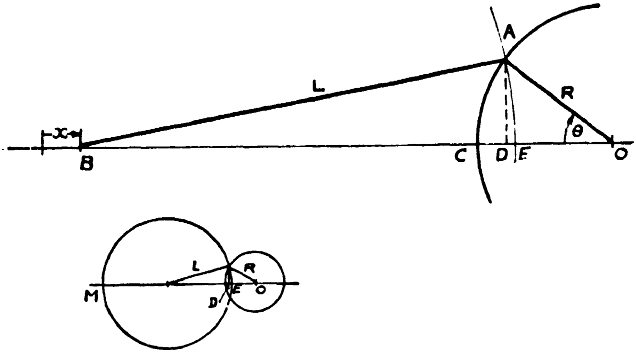
Now from Fig. 74*a* we see that

$$MD \,.\, DE = AD^2 = R^2 \sin^2\theta,$$

or

$$DE = \frac{R^2 \sin^2\theta}{MD},$$

and provided the ratio $L/R = n$ is large (say $\geqslant 4$) we shall not be far wrong in assuming that $MD = ME = 2L$.



(a)

FIG. 74.

Thus *approximately*

$$x = R(1 - \cos\theta) + \frac{R \sin^2\theta}{2n};$$

$$\therefore \frac{dx}{dt} = R\left[\sin\theta \,.\, \frac{d\theta}{dt} + \frac{2 \sin\theta \,.\, \cos\theta}{2n} \cdot \frac{d\theta}{dt}\right].$$

But $d\theta/dt =$ the angular velocity of the crank which is assumed to be constant $= \omega$.

Hence

$$\frac{dx}{dt} = R\omega\left[\sin\theta + \frac{\sin 2\theta}{2n}\right],$$

and, differentiating again,

$$\frac{d^2x}{dt^2} = R\omega^2\left[\cos\theta + \frac{\cos 2\theta}{n}\right].$$

The first term of the above expression represents the simple harmonic acceleration of the reciprocating parts, which would occur with an infinite connecting rod, while the second shows the modification introduced by having a finite rod.

If we had not made the approximation stated, an even cosine series would have been obtained as the correct expression:

$$R\omega^2 [\cos \theta + A_2 \cos 2\theta + A_4 \cos 4\theta + \ldots],$$

where A_2, $A_4 \ldots$ are constants involving powers of $\frac{1}{n}$ which, however, become so small after the second term that for all practical purposes they may be neglected, except perhaps for certain high speed short connecting rod engines.

The values for $n = 4{\cdot}5$ are as follows:

$$A_2 = 0{\cdot}225,\ A_4 = 0{\cdot}0028,\ A_6 = 0{\cdot}00004.$$

The force P_f required to accelerate the reciprocating parts is $\frac{W}{g} \times$ (above expression), and the corresponding reaction on the engine frame consists therefore of harmonic components of frequency 1, 2, 4, 6 ... times engine speed, all acting in the line of stroke.

If P = the fluid thrust in the cylinder, the force transmitted in the line of stroke is then $P - P_f = P_0$ (Fig. 75). Now the gudgeon pin is in equilibrium under the action of three forces (*a*) P_0, (*b*) the side thrust F, and (*c*) the thrust P_r of connecting rod on the pin, or in other words the thrust transmitted along the rod is the resultant of P_0 and F, i.e.

$$P_r = \frac{P_0}{\cos \alpha}.$$

If P_c = the crank effort, i.e. the normal force acting at the crankpin, the turning moment $= P_c \times R$, and from the figure it will be seen that

$$P_c = P_r \cos [90^\circ - (\alpha + \theta)] = P_r \sin (\alpha + \theta).$$

But $$AD = L \sin \alpha = R \sin \theta.$$

Hence α is known in terms of the crank angle θ

Alternatively, we may obtain the turning moment graphically :

At N, the point where the connecting rod or the connecting rod produced cuts the perpendicular to the stroke line through

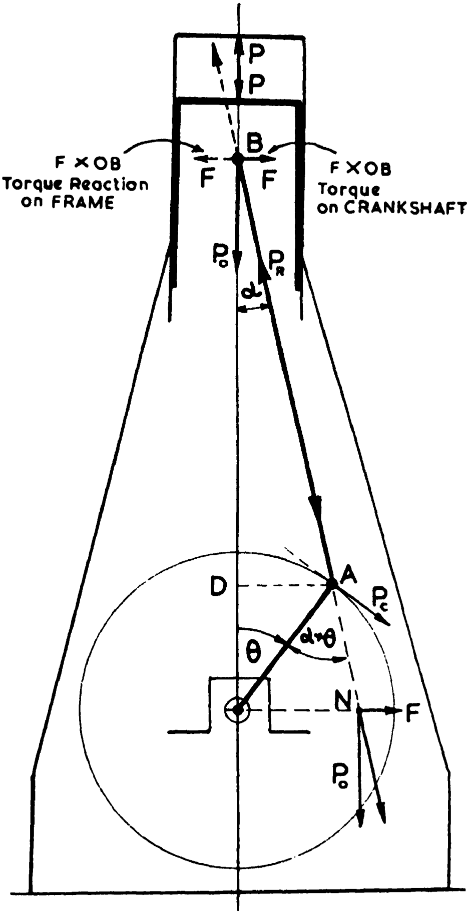


FIG. 75.

O, let the thrust along the rod P_r be split up into its two components P_0 and F. The torque exerted is then seen to be

$$P_0 \times ON, \text{ since } F \text{ has no turning effect about } O.$$

This is the best method to adopt for constructing a turning-moment diagram for an engine.

Again, considering the components of P_r at the gudgeon pin B, the torque on the crankshaft could be expressed as

$$F \times BO$$

at any instant, since P_0 has no moment about O.

Now, besides the forced torsional oscillations of the crankshaft, the torque fluctuations will produce a rocking vibration of the engine on its foundations. This latter effect we may consider in two ways, either from the fact that the torque supplied by the engine to rotate the shaft must be accompanied by an equal and opposite torque reaction tending to produce rotation of the engine round the shaft centre, e.g. if the crank were fixed and the engine free, the latter would rotate bodily about the crankshaft; or, alternatively, this torque reaction is produced by the side thrust on the engine frame of the horizontal component F of the thrust of the connecting rod *on* the piston or crosshead (shown dotted Fig. 75), which causes the oscillation of the frame about O.

Harmonic orders. The torque curve for any cylinder may be split up into harmonic components, the frequencies of which can be represented in terms of the engine speed, and *the number of cycles performed per revolution of the engine* is termed the *order* number. For instance:

4-*Stroke Engine.* The fundamental torque fluctuation occurs in two revolutions of the engine, hence there will be $\frac{1}{2}$ cycle per revolution, i.e. we have a torque fluctuation of order $\frac{1}{2}$.

The 3rd harmonic (say) having a frequency three times the fundamental would be denoted as of order $1\frac{1}{2}$.

Thus, the component harmonics for a 4-stroke engine can be of every integral order and half order as well.

2-*Stroke Engine.* The fundamental torque fluctuation occurs in one revolution, hence the harmonic components will complete 1, 2, 3 ... cycles per revolution and will therefore be distinguished by order numbers 1, 2, 3 ... .

Let us examine the effect of these torque variations on the crankshaft system.

Forced torsional oscillations of the shaft systems of multicylinder engines. Critical speeds. As early as 1900 Dr. Bauer showed that dangerous synchronous oscillations could occur in marine steam engine crankshafts, and from that time to the present many cases of shafting failure have been traced to this cause.

We have seen that the shafting, being an elastic system, possesses a series of modes of natural vibration distinguished by the number of nodes. Hence, if the engine is run at such a speed that the frequency of any harmonic component of the torque curve of any one cylinder coincides with that of one of these natural modes, resonance will occur and dangerous amplitudes may be forced.

Such engine speeds are called *critical speeds* and are usually denoted by two quantities: (*a*) the number of nodes of the natural vibration with the frequency of which the forced coincides, and (*b*) the *order number* of the harmonic causing resonance, e.g.:

In a 4-stroke engine we might have (say) a critical speed of 2 nodes and order $3\frac{1}{2}$, i.e. the 7th harmonic of the torque curve, causing $3\frac{1}{2}$ vibrations per revolution, is coinciding with the 2-noded natural mode.

Thus a succession of criticals will exist for *each* mode of natural vibration of the shaft, occurring at engine speeds

$$=\frac{\text{Natural Frequency}}{\text{Order Number}},$$

e.g. suppose the 1-node natural frequency is 400 cycles per *minute*, critical engine speeds will occur as follows for a 4-stroke engine:

Harmonic	*Order No.*	*Critical Engine Speed*
1	½	800 r.p.m.
2	1	400
3	1½	267
4	2	200
5	2½	160
6	3	135
7	3½	114·5
8	4	100
9	4½	89
10	5	80
11	5½	72·5
12	6	66·7

etc.

If the above system was that of a marine installation which would operate over a range of speed, with (say) 100 r.p.m. as mean, it can be seen how difficult it would be to avoid a critical.

Torque phase diagrams. Although we have stated that criticals can exist corresponding to any harmonic of any one cylinder, it will be evident that the resultant effect produced on the shafting as a whole by all the cylinders will be large or small depending on how nearly the corresponding harmonics from each cylinder act together or come to neutralising each other, i.e. depending on the phase relationship of the respective harmonics of each cylinder.

The harmonic torques can, as usual, be represented by rotating vectors and will come into operation in the same sequence as the firing order, one revolution of the vector giving one complete fluctuation of the harmonic concerned, so that the speed of the vector will be equal to the engine speed multiplied by the order number.

Consider a 4-cylinder, 4-stroke compression-ignition engine: Fig. 76 shows the crank arrangement, firing order and torque curve for one cylinder. The first two harmonic components of the torque curve are also shown, i.e. for orders $\frac{1}{2}$ and 1, the generating vectors therefore revolving at $\frac{1}{2}$ engine speed and engine speed respectively.

Assuming that the fuel injection arrangements are the same for each cylinder, the torque curves and their harmonics can be considered to be the same for each cylinder.

The vectors generating the various harmonics are all at different phase angles at any given instant, but we are concerned only with the phase relationship of the vectors of the *same* harmonic in each cylinder, and each of these vectors will always be in the same position at the same point in the cycle. For instance, the position of the $\frac{1}{2}$ order vector for cylinder (1) is shown in Fig. 76 when the crank is at the inner Dead Centre just starting on the firing stroke, and the $\frac{1}{2}$ order vectors for the other cylinders will be in a similar position when their respective cranks are just starting on the firing stroke. In arriving at their phase relationship we can simplify matters by turning the vectors all through the same angle such that they will coincide with the crank at (say) the I.D.C. on the firing stroke. Thus :

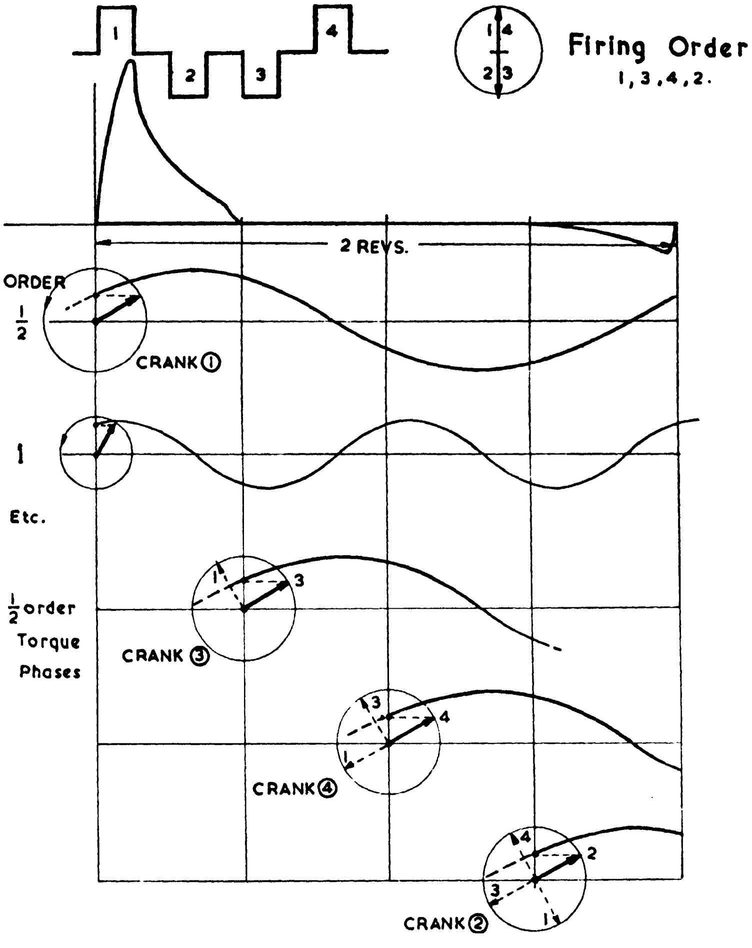


FIG. 76.

Order $\frac{1}{2}$. If crank (1) is at the I.D.C. on the firing stroke,
Torque vector (1) is assumed to coincide with it (Fig. 77*a*)

In $\frac{1}{2}$ revolution of the shaft, crank (3) arrives at the I.D.C. about to start the firing stroke :

Crank (1) has turned through 180°,
Torque vector (1) has turned through 90°, i.e.

Torque vector (3) is 90° behind vector (1) (Fig. 77*b*).

In another ½ revolution of the shaft, cylinder (4) fires: i.e.

Torque vector (4) is 90° behind vector (3) (Fig. 77*c*),

and similarly,

Torque vector (2) is 90° behind vector (4).

The complete phase diagram is therefore as in Fig. 77*d*, and has also been shown in Fig. 76.

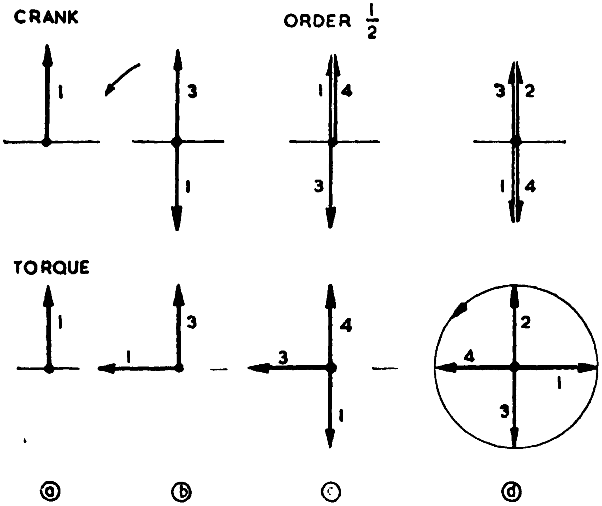


FIG. 77.

Order 1. The Torque vectors revolve at crank speed, i.e. the vectors lag behind vector (1) by twice the angle for order ½ (Fig. 78).

Similarly for

Order 1½. The vectors lag behind vector (1) by three times their angle for order ½, and for

Order 2. The vectors lag behind (1) by four times the angle for order ½.

For higher orders the vector diagrams now repeat, and it will be found, for the case considered, that the torque vectors for orders 2, 4, 6 ... *all act in phase.*

Similarly, for any engine with equal firing intervals, the vectors act in phase for all orders which are *equal to, or a multiple of, the number of firing impulses per revolution of the engine.*

Such orders are called *Major orders*, and the corresponding critical engine speeds are similarly termed Major Criticals, e.g. suppose that the 1-node natural frequency was 1,200 vibrations per minute, then

the 4th harmonic (order 2) coincides at engine speed of 600 r.p.m.,

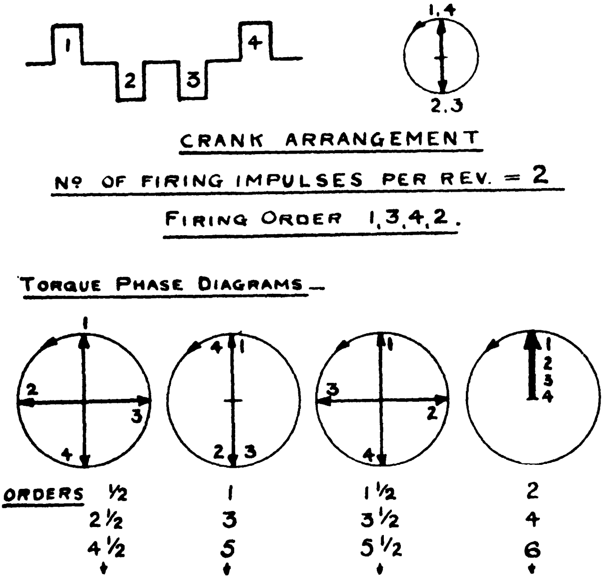


FIG. 78.

and the 8th harmonic (order 4) coincides at engine speed of 300 r.p.m., etc.

The other orders are called *Minor orders*, but cannot necessarily be disregarded, indeed, as we shall see, they may be more serious than the major orders. Although, at first sight, the torque vectors would appear to neutralise, the resultant effect produced on the shaft depends not only on the phase relationship of the torques, *but also on the point of application of the various torques along the shaft*, since the further away the point of application is from a node the more effect will the torque produce on the system.

Resultant effect due to a given harmonic torque acting at each cylinder of an engine. Amplitude vector diagrams. We have already seen how the amplitude of vibration at any point of a shaft system can be determined when a simple harmonic torque, *of frequency other than the natural,* is applied. The curve of angular displacement is *not,* of course, the normal elastic curve. If this same harmonic is applied in turn at each cylinder of a multi-cylinder engine, the corresponding amplitudes can similarly be calculated (see tables I and II, page 191), and the resultant effect at any cylinder will be the vector sum of the amplitudes taken in the phase relationship of the torques since, without damping, the displacement vectors are all in phase (or 180 degrees out) with their torque vectors.

Under *resonant conditions* we know that the amplitude of a forced vibration, without damping, will become infinite, but in practice some damping is usually present which will keep the amplitude finite, and the resultant effect from all the cylinders will depend on the resultant energy input.

To investigate the matter, let us make some simplifications and assume that a single harmonic torque acts on the system at one of the cylinders, whilst damping, varying as the velocity, is concentrated at another load point. We will also assume that damping does not alter the *form* of the vibration, i.e. all the loads will still reach their maximum amplitudes together *and follow the normal elastic curve for the natural vibration.*

Let the torque act on the *m*th load, the motion of this load being

$$\theta_m = a_m \cos qt,$$

so that the torque, being 90 degrees ahead in resonance, will be

$$-T_m \sin qt.$$

Let the damping act at the *n*th load, the motion of which being

$$\theta_n = a_n \cos qt,$$

so that the resisting torque will be

$$-T' \frac{d\theta_n}{dt} = T' a_n q \,.\, \sin qt,$$

where T' = the frictional resisting torque per radian per sec.

Now the work done by the disturbing torque in one cycle must be equal to the energy absorbed at the damping point.

We have already seen on page 163 that only the component of a harmonic torque at right angles to the displacement vector does work in a complete cycle. Here, in resonance, the torque vector itself is 90 degrees in advance so that the work done is

$$\int_0^{2\pi}(-T_m \sin qt)\frac{d\theta_m}{dt}\cdot dt = T_m a_m \int_0^{qt=2\pi} \sin^2 qt\, d(qt)$$

$$= T_m a_m \pi.$$

Similarly the damping torque $T'a_n q \sin qt$ is generated by a vector $T'a_n q$ at right angles to the displacement vector so that by substituting this value in place of T_m above the energy absorbed is

$$(T'qa_n)\, a_n\pi = T'qa_n^2\pi.$$

[If $m=n$, we get $T_m = T'aq$ corresponding to $P=F_1Aq$ before.]

Thus, equating the above two expressions,

$$a_n = \frac{T_m}{T'q}\cdot\frac{a_m}{a_n}.$$

The *ratio* a_m/a_n is known from the normal elastic curve $=\frac{\lambda_m}{\lambda_n}$, so that a_n is known; or, expressing things in terms of a_1, the amplitude at cylinder (1):

$$a_1 = a_n\cdot\frac{a_1}{a_n} = a_n\cdot\frac{\lambda_1}{\lambda_n} = \frac{T_m}{T'q}\cdot\frac{\lambda_m}{\lambda_n}\cdot\frac{\lambda_1}{\lambda_n},$$

where the λ-ratios are obtained from the N.E.C.

As an example, let the four-cylinder engine, for which the torque phase diagrams have been drawn, be coupled to a flywheel (5) and propeller (6). We will assume that the portion of the N.E.C. for the engine masses for the mode of natural vibration concerned is as shown in Fig. 79*a*.

Consider (say) the ½ order harmonic torque from cylinder (1), of maximum value T_1, to be forcing at that cylinder a vibration $a_{11} \cos qt$, the generating vector for which will, in resonance, be 90 degrees behind the torque vector. These vectors are shown in Fig. 79*b*, and if YY' is the projection line, the load at the instant shown is at its maximum displacement a_{11} (the first suffix shows the load concerned, and the second the torque position—here at cylinder (1)).

The loads at the other cylinder points will also be at their maximum displacements, the amplitudes corresponding to the N.E.C., namely a_{21}, a_{31}, a_{41} (Fig. 79c), where

$$a_{21} = a_{11} \cdot \frac{\lambda_2}{\lambda_1}, \text{ etc.}$$

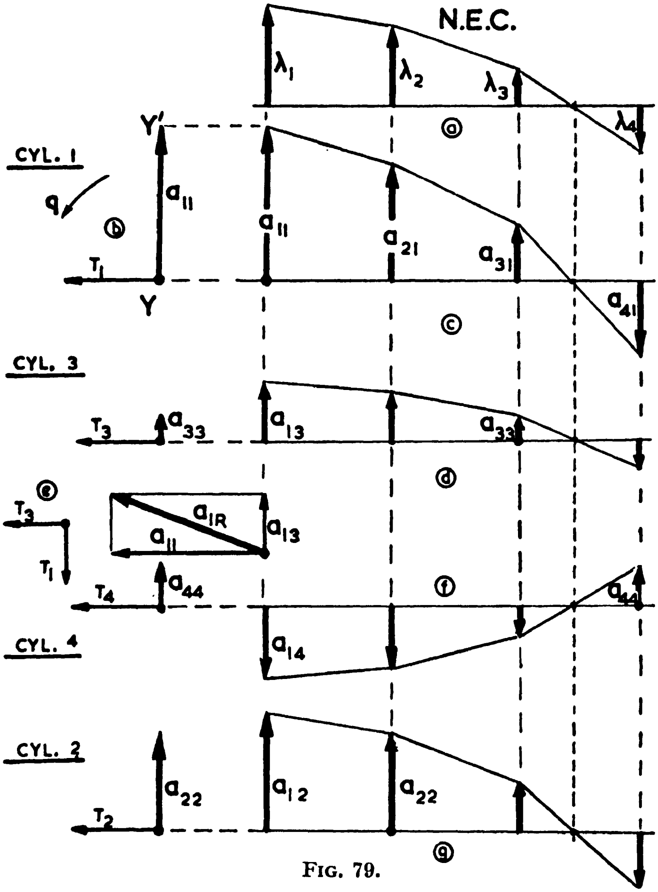


FIG. 79.

Now consider the next torque in phase sequence, i.e. $-T_3 \sin qt$ from cylinder (3) forcing a vibration alone. This vibration will reach a maximum 90 degrees after the vibration set up by T_1 (Fig. 79*d*), the amplitude being a_{33} at cylinder (3), while the amplitudes at the other cylinders are proportional to the same N.E.C.

If these two vibrations due to T_1 and T_3 are combined the resultant will also be similar to the N.E.C., the amplitude at cylinder (1) being a_{1r} (Fig. 79*e*).

In another 90 degrees of vector rotation the forced vibration, due to $-T_4 \sin qt$ acting alone, reaches its maximum a_{44} at cylinder (4); and similarly in another 90 degrees the forced vibration due to $-T_2 \sin qt$ reaches its maximum (Figs. 79*f*, *g*).

If the damping is entirely due to the propeller action in the water, i.e. is concentrated at load (6), the amplitude forced at cylinder (1) will therefore be as follows:

$$\left.\begin{array}{rl}\text{Torque at load (1)} & m=1\\ \text{Damping at load (6)} & n=6\end{array}\right\} \quad \therefore\ a_{11}=\frac{T_1}{T'q}\left(\frac{\lambda_1}{\lambda_6}\right)\left(\frac{\lambda_1}{\lambda_6}\right).$$

$$\left.\begin{array}{rl}\text{Torque at load (2)} & m=2\\ \text{Damping at load (6)} & n=6\end{array}\right\} \quad \therefore\ a_{12}=\frac{T_2}{T'q}\left(\frac{\lambda_2}{\lambda_6}\right)\left(\frac{\lambda_1}{\lambda_6}\right).$$

$$\text{Similarly}\quad a_{13}=\frac{T_3}{T'q}\left(\frac{\lambda_3}{\lambda_6}\right)\left(\frac{\lambda_1}{\lambda_6}\right),$$

$$\text{and}\quad a_{14}=\frac{T_4}{T'q}\left(\frac{\lambda_4}{\lambda_6}\right)\left(\frac{\lambda_1}{\lambda_6}\right).$$

The resultant amplitude of load (1) will be the vector sum of the respective amplitudes due to the individual torques, i.e. $a_1' = \Sigma(a_{11}+a_{12}+a_{13}+a_{14})$, where the phase difference of these vectors is exactly the same as that of the torques which produce them (but lagging by 90 degrees).

$$\text{Thus}\qquad a_1'=\frac{1}{T'q}\left(\frac{\lambda_1}{\lambda_6}\right)\Sigma\frac{T_m\lambda_m}{\lambda_6},$$

where $\frac{\lambda_m}{\lambda_6}$ is the ratio of the amplitude at the cylinder at which the harmonic torque T_m acts and the amplitude at the damping point.

We have, however, assumed that T_m is the same for each cylinder $=T$, so that

$$a_1'=\frac{T}{T'q}\,\frac{\lambda_1}{\lambda_6}\cdot\frac{\Sigma\lambda_m}{\lambda_6},$$

where $\Sigma\lambda_m$ is the vector sum of the ordinates of the N.E.C.

While the determination of the actual amplitudes may be difficult due to the uncertainty of the amount of the damping and where it is applied, yet the above investigation will enable us to estimate whether a critical is likely to be dangerous or not, i.e. whether the resultant amount of energy added in a cycle is great or small, *since the resultant amplitude will be proportional to the vector sum of the ordinates of the N.E.C. taken in the same phases as their corresponding torques.*

In the example given it has been assumed that a node occurs between cylinders (3) and (4), so that, when taking the vector sum, the amplitude vector a_{14} for cylinder torque (4) will be drawn in the reverse direction to torque vector T_4—see Figs. 79*f* and 80.

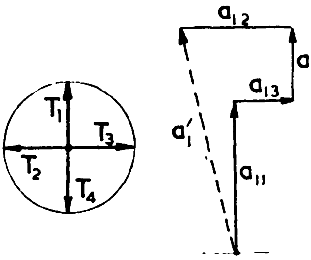


FIG. 80.

In drawing these vectors there is no need, of course, to consider the 90 degrees of lag with the respective torque vectors—they may be drawn parallel to the respective torque vectors and in the same or opposite sense according to the side of the node on which they are situated.

The position of a node is one of the main factors in determining the importance of a critical.

For instance, suppose that the node is a long way from the engine, in which case the ordinates of the N.E.C. at the cylinder points $\lambda_1, \lambda_2 \ldots$ will be approximately equal, so that, taking their vector sum in the phases of the torques, the minor orders will neutralise and only the majors will have effect (Fig. 81*a*).

But suppose a node was situated mid-way between cylinders (2) and (3) such that $\lambda_1 = -\lambda_4$ and $\lambda_2 = -\lambda_3$, then it will be seen (Fig. 81*b*) that the *majors will have zero effect* and the minors may be important.

Total motion of the shaft system. It should be pointed out that in the previous investigations we have considered the effect of the *same* harmonic at each cylinder, but *all* the harmonics of the torque curve are in operation together, each forcing its own vibration, resonant or non-resonant, as above described, so that the true total effect at any instant will be equal to the sum of the simple harmonic motions generated by vectors revolving at 1, 2, 3 ... times the fundamental.

Although we have been considering exact coincidence of the forced harmonic torque frequency and the natural frequency of the shaft system, it will be remembered from our previous remarks that dangerous amplitudes are possible over a fairly wide range on either side of the exact critical speed and due allowance must be made for this possibility in the design.

It will be evident that with a multi-cylinder Diesel or other variable speed internal combustion engine it may be extremely difficult to avoid vibration due to the large range of criticals that will exist.

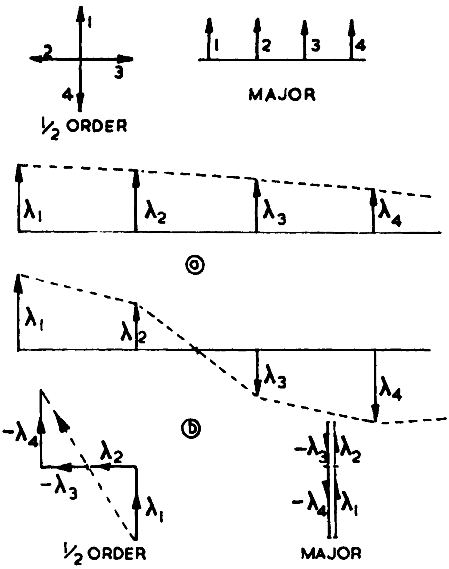


FIG. 81.

In general, a periodic disturbing force producing synchronous oscillation in an elastic structure makes its presence known by the unwelcome vibrations transmitted to the senses, but in the case of a shaft undergoing torsional oscillation no such outward sign of anything untoward may be noticeable, it being possible for large amplitudes and dangerous stresses to be produced without any indication of their existence, since if the loads consist of uniform disks, etc., their centres of gravity will be on the axis of rotation. With an engine crankshaft a certain amount of external vibration may be evident due to the reciprocating masses.

In some cases, only on fracture occurring has it been discovered that a system must have been in torsional oscillation, whilst in others a critical has been detected by the portion of the shaft at which the node was formed warming up due to elastic hysteresis.

EXAMPLES IV

1. The natural frequency of one of the modes of torsional oscillation of the shaft system of a 4-cylinder, 4-stroke I.C. engine is 1,000 cycles per minute, and the relative amplitudes at the cylinders taken in order are 1, 0·75, 0·5, and $-0{\cdot}5$. The maximum values (T) of the harmonic torques relative to the fundamental are given in the table and may be assumed to be the same for each cylinder.

Harmonic	T
Fundamental	1
Second	1·2
Third	0·8
Fourth	0·6

Determine (*a*) the critical engine speeds up to the first major; (*b*) the relative importance of each critical in regard to the forcing of dangerous amplitudes. [Liv. U.]

2. Explain the meaning of major and minor critical speeds of an engine installation.

The following data refer to a six-cylinder, four-stroke I.C. engine, the cylinders being numbered in order:

Cranks arranged in pairs at 120°: 1, 6; 2, 5; 3, 4.
Firing order: 1, 5, 3, 6, 2, 4.

Investigate the relative importance of the 2-node criticals of orders 1, $1\frac{1}{2}$, and 3, for which the maximum values of the corresponding harmonic torques are in the ratio of 1 : 0·8 : 0·5, and the ordinates of the normal elastic curve at the respective cylinders are (starting with cylinder 1) 1·0; 0·75; 0·5; 0·2; $-0{\cdot}3$, $-0{\cdot}6$. [Liv. U.]

CHAPTER X

ELIMINATION AND ABSORPTION OF VIBRATION

WHILE the *free natural vibrations* of an elastic system will always in time be damped out, it is interesting to note that certain arrangements of a given system will continue to vibrate for a longer time than others.

It must be realised that, since no external forces are acting when a system is in free vibration, *the position of the centre of gravity of the complete system must remain unchanged.* Thus, if we take the simple case of a load supported by a spring, as the load moves downward the foundation framework must move upward, i.e. vibrations are produced in the foundations, the magnitude of which will, of course, depend on their mass. It is really only the case of two masses in free vibration at the ends of an elastic rod or other coupling. When in vibration a node is formed between them, and if one mass is made infinitely large, the node to all intents coincides with this mass, which therefore may be regarded as a " rigid " support.

Consider then a load *W* at the end of an elastic beam which is firmly fixed at the other end to the ground (Fig. 82*a*). Any movement of *W* in one direction must be accompanied by a movement of the foundation in the opposite direction during a free vibration. The movement of the foundation will obviously be quickly damped out by friction hence the motion of *W* will also quickly cease.

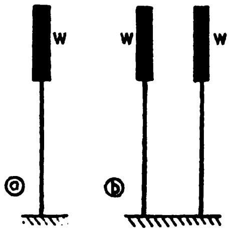


FIG. 82.

But suppose we arrange two exactly similar systems side by side as shown (*b*) and set the masses moving in opposite directions, then no motion of the foundations will take place, and, as there will be no frictional damping from this cause, the masses will continue to vibrate for a longer time than each would singly.

If the masses were started in the *same* direction the motion would soon cease and become as above, since any motion which sets up a movement of the foundations will rapidly be damped out. This is the principle of the *tuning fork.*

Twin marine Diesel engines placed side by side constitute an exactly similar system and often have their heads clamped together by crossbars to prevent this tuning fork vibration, which may easily be set up by torque reaction, especially if the foundations are relatively flexible as might be the case on board a ship.

There are certain main methods by which we can set about the problem of eliminating or reducing vibration in a structure :

(1) Removing the disturbing force.
(2) Reducing the effect of the disturbing force which is transmitted to the foundations.
(3) Introducing an equal and opposite disturbance.
(4) Avoiding resonant conditions.
(5) Increasing the damping.

Let us examine these in turn.

(1) **Removal of the disturbing force.** If the disturbing force is caused by unbalanced rotating masses, this can and always should be eliminated by the addition of two extra masses in separate parallel planes of rotation. For instance, suppose we have a shaft subjected to a series of centrifugal forces set up by masses whose centres of gravity do not coincide with the axis of rotation.

In Fig. 83 two such forces P and Q are shown. We can obviously replace these forces by the parallel forces P_1, P_2 and Q_1, Q_2 in any two planes of revolution A and B as shown, where

$$P_1 + P_2 = P,$$

and
$$P_1 a = P_2 b,$$

and similarly for Q_1 and Q_2.

P_1 and Q_1 can now be combined into a single resultant force R_1, which can be neutralised by placing a mass W_1 in plane A so as to set up an equal and opposite centrifugal force. Similarly a mass W_2 in plane B will balance the resultant R_2 of P_2 and Q_2. A complete discussion on Balancing is outside the scope of this work, and the reader is referred to *The Balancing of Engines* by Professor W. E. Dalby, or text-books on the Theory of Machines.

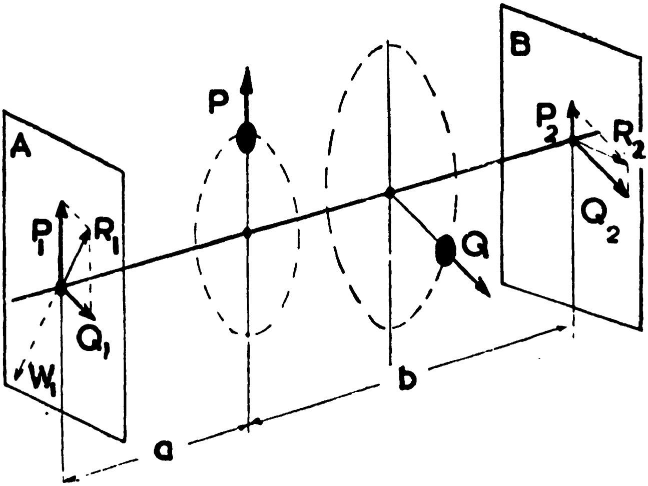


FIG. 83.

The disturbing effect due to the inertia reaction of the reciprocating parts of a multi-cylinder engine can be eliminated or reduced by suitably arranging the crank angles and the magnitudes of the reciprocating masses. Such balancing is done in large 4-crank triple or quadruple expansion marine engines. With Diesel or petrol engines it is usual to have the cranks at equal intervals with equal reciprocating masses. We have seen that the inertia reaction in the line of stroke for any cylinder is given by an even cosine series :

$$P_f = \frac{W}{g}\,\omega^2 R(\cos\theta + A_2\cos 2\theta + A_4\cos 4\theta + \ldots)$$

where θ is the angle turned through by the crank from the inner dead centre. Hence these harmonics may be represented by vectors rotating at 1, 2, 4 ... times engine speed. The respective vectors for any cylinder will all coincide when the crank is at the inner dead centre ($\theta = 0$), and will have turned through

1, 2, 4... times the actual crank angle from the I.D.C. at any other instant.

Perfect inertia balance is usually taken to mean that first and second harmonics neutralise, but if we draw the vector diagrams for a multi-cylinder engine we shall find that, with equal crank angles, the harmonics which are an even multiple of half the number of cylinders are all in phase. Fig. 84 shows the diagrams for a 4- and a 6-cylinder engine, crank (1) being at the I.D.C.

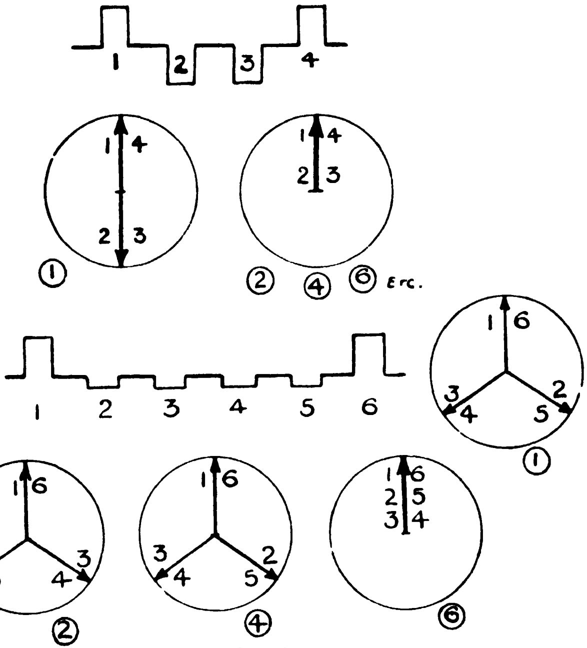


FIG. 84.

The forces represented by the projection of these vectors all act in the plane of reciprocation, and besides causing a periodic disturbance in this plane also tend to produce rotation of the engine frame about an axis through the centre of gravity perpendicular to this plane. If, however, we assume that the engine is symmetrical on either side of the plane containing the C.G. and at right angles to the shaft, then with equal reciprocating masses per cylinder it will be clear that the couples introduced will balance if the cranks are arranged with "lookingglass symmetry" in regard to this central plane, i.e. one half of the shaft is a reflexion of the other (Fig. 85).

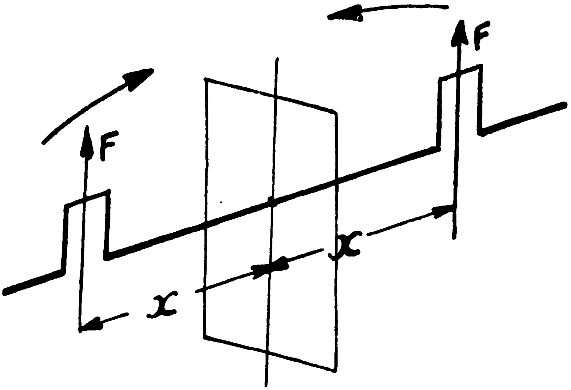


FIG. 85.

(2) **Reducing the effect of a disturbing force.** We have noted that if a periodic disturbing force acts on an elastic system, its effect will be transmitted via the foundations to every part of the building or surrounding structures, any portion of which may be set into unwelcome vibration if its natural frequency coincides with that of the disturbance.

It has been shown that the amplitude of a single load undergoing a forced vibration is $\frac{n^2}{n^2-q^2}$ or $\frac{1}{1-q^2/n^2}$ times the displacement that would be produced if the disturbing force was applied statically.

Now upon the amplitude depends the extension or compression of the elastic connexion and hence the force which will be transmitted to the foundations. If we design this elastic support so that the natural frequency of the load is small compared with the forced frequency, then, as we have already seen, the forced amplitude can be reduced almost to zero, the load remaining approximately stationary in space, and the more nearly we can approach these conditions the less will be the force received by the foundations.

The magnification factor becomes -1 when

$$\left(\frac{q}{n}\right)^2=2,$$

hence the frequency ratio must be greater than $\sqrt{2}$ in order to cause a *reduction in amplitude* over that produced when the disturbing force is statically applied. For instance, if the frequency ratio was 5 then the magnification factor would be

$$\frac{1}{1-25}=-\frac{1}{24},$$

and the force on the foundations would only be $\frac{1}{24}$ of the actual disturbing force acting.

This is all very well in theory, but it will be remembered (page 205) that a difficulty arises in practice from the fact that an elastic system must have a very large static deflexion under the given load to give a small natural frequency. This means that we should require a great amount of space to house an elastic system that would be strong enough to support the load and yet have the required flexibility.

An interesting example is cited by Dr. T. W. F. Brown in his paper " Vibration Problems from the Marine Engineering point of view "* as follows :

4-cylinder, 2-stroke I.C. engine, 2,500 H.P. at 110 r.p.m.
Weight 190 tons.

The fundamental torque fluctuation occurs at 110 r.p.m., and the second order at 220 r.p.m.

Second order forces would be large on such an engine, and are therefore suitable for reduction.

Assuming that the supports are to be designed for a natural frequency of $\frac{220}{3}$ r.p.m., the reduction in transmissibility

$$=\frac{1}{1-(3)^2}=(-)\frac{1}{8}.$$

$$\text{Natural frequency}=\frac{220}{3\times 60}=\frac{1}{2\pi}\sqrt{\frac{32\times 12}{\delta_{st}}};$$

$$\therefore\ \delta_{st}=6{\cdot}5\text{ in.}$$

Thus the springs would have to deflect 6·5 in. under a load of 190 tons.

Taking 10 springs on each side of the engine base, the load on each spring will be 9·5 tons.

Suitable springs would be of a square section of 3·25 in. side,

---

* North East Coast Institution of Engineers and Shipbuilders. 1939.

18 in. mean coil diameter, and 14·5 coils, but their great bulk would be an insuperable difficulty in this case.

A stiffer and therefore less bulky system could, of course, be employed to eliminate a harmonic of the actual disturbance which may be causing resonant conditions in some part of the building, etc.

Steel springs are used whenever possible, but rubber or cork pads between an engine bed and the foundation have been employed quite effectively as a stiffer absorber for a harmonic, the only defect being that such substances are liable to deterioration.

Whatever method of mounting is employed, there must be no connexion with the foundations other than by the elastic system.

Many petrol and Diesel road vehicle engines are now mounted on flexible connexions for this reason—either rubber pads or flat flexible steel plates, whilst the tendency of the engine to vibrate with torque reaction fluctuations is often discouraged by frictional damping rings, which are heavy metal rings encircling and frictionally coupled to the engine block, but entirely free from the chassis.

Transmissibility.

$$\text{The ratio } \frac{\text{Force transmitted to the foundations}}{\text{Disturbing force } P}$$

is called the *Transmissibility.*

(*a*) *No damping*:

$$\frac{W}{g}\frac{d^2x}{dt^2}+Fx=P\cos qt.$$

We have seen that the

$$\textit{magnification factor}=\frac{\text{Actual amplitude}}{\text{Static displacement } P/F}$$

$$=\frac{\text{Actual amplitude}\times F}{P}=\frac{\text{Actual force in springs}}{\text{Maximum disturbing force}}$$

$$=\frac{n^2}{n^2-q^2}=\frac{1}{1-q^2/n^2}.$$

In this case the force transmitted to the foundations is entirely due to the springs and the transmissibility is the same as the magnification factor.

To reduce the force transmitted, the frequency ratio q/n must be greater than $\sqrt{2}$, or, alternatively, if q is less than $\sqrt{2}\,.\,n$ the springs would make matters worse since the magnification factor is greater than unity.

(*b*) *With damping* :

$$\frac{W}{g}\frac{d^2x}{dt^2}+F_1\frac{dx}{dt}+Fx=P\cos qt.$$

In this case the transmissibility is *not* the same as the magnification factor if damping is equivalent to a dashpot, since besides the spring force (on which the magnification factor depends) there is now the force transmitted to the ground via the dashpot.

Referring to the vector diagram (Fig. 52), we have

$$\text{Maximum spring force}=oa=FA,$$

$$\text{Maximum force to overcome damping}=od=F_1qA.$$

These vectors are at 90 degrees to one another, *and their resultant is now the maximum force to which the foundations are subjected*, i.e.

$$\text{Maximum force transmitted}=A\sqrt{F^2+(F_1q)^2}.$$

$$\text{But}\quad A=\frac{f}{\sqrt{(n^2-q^2)^2+4k^2q^2}}=\frac{f}{n^2}\times\frac{1}{\sqrt{\left(1-\frac{q^2}{n^2}\right)^2+K^2\cdot\frac{q^2}{n^2}}}$$

where $K^2=4k^2/n^2$

$$=\frac{f}{n^2}\times\text{Magnification factor}$$

$$=\frac{P}{F}\times\text{Magnification factor.}$$

Thus the Transmissibility

$$=\frac{\text{Actual force}}{P}=\frac{A\sqrt{F^2+(F_1q)^2}}{P}$$

$$=\frac{\text{Magnification factor}}{F}\times\sqrt{F^2+(F_1q)^2}$$

$$=\text{Magnification factor}\times\sqrt{1+\left(\frac{F_1}{F}\cdot q\right)^2}$$

$$=\text{Magnification factor}\times\sqrt{1+K^2\cdot\frac{q^2}{n^2}}.$$

Fig. 86 shows the variation of transmissibility with frequency ratio q/n for different values of K^2, i.e. for a given spring system with different dampings, and the difference between these curves and those for the magnification factor should be carefully noted, particularly in the region $q/n > \sqrt{2}$. *It will be seen that in this region increase of damping causes a slight increase in the transmissibility.*

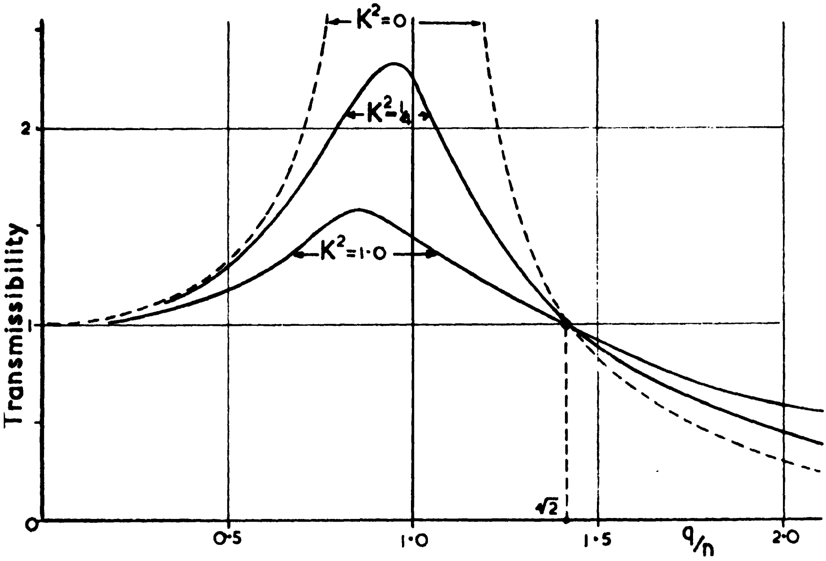


FIG. 86.

Example.

A single load supported by an elastic structure is subjected to a simple harmonic disturbing force, whilst damping, varying as the velocity of the load, is effected by means of a dashpot. For such a system it is found that the magnification factor is a maximum when the frequency of the disturbing force is $\frac{\sqrt{3}}{2}$ times the natural frequency. Determine the transmissibility under these conditions.

[Liv. U.]

Ans. 1·764.

(3) The undamped vibration absorber. If we cannot neutralise the disturbing force at the source by balancing, it is sometimes possible to cause the amplitude of the vibrating system to become zero by introducing a small auxiliary vibrating system.

To understand the principle of this interesting device, the reader is asked to refer to the problem on page 180 illustrating the application of a simple harmonic force on a two-mass system.

The amplitude of the load W_1 upon which the disturbing force acts was found to be

$$y_1 = \frac{-P\left(\frac{W_2}{g} q^2 - F_2\right)}{\left[\frac{W_1}{g} q^2 - (F_1 + F_2)\right]\left(\frac{W_2}{g} q^2 - F_2\right) - F_2^2}.$$

This will become *zero* when $\left(\frac{W_2}{g} q^2 - F_2\right) = 0$,

i.e. when $$q^2 = \frac{F_2 g}{W_2} \quad \text{or} \quad \frac{q}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{F_2 g}{W_2}},$$

which is the natural frequency of the load W_2 on the spring F_2 alone.

Thus, if we have a machine W_1 on an elastic foundation F_1 which is emitting a disturbing force $P \cos qt$, the effect of this in forcing unwelcome vibrations may be neutralised by attaching an auxiliary vibratory system W_2 on a spring F_2, *of frequency equal to that of the disturbance, thus bringing the machine* W_1 *to rest.*

Substituting $y_1 = 0$ in equation (1) or $\left(\frac{W_2}{g} q^2 - F_2\right) = 0$ in the expression for y_2 we see that

$$F_2 y_2 = -P \quad \text{or} \quad F_2 x_2 = -P \cos qt,$$

which means, as we might expect, that the force exerted in the spring F_2 at any instant is equal and opposite to the disturbing force.*

Difficulties may arise in practice due to the impossibility in certain cases of obtaining satisfactory dimensions for the attached load and its elastic coupling. It will also be evident that this method can only be used for the elimination of the effect of a *constant frequency* disturbing force, and by making a two-mass system out of the original one-mass system there is now introduced another speed of the machine at which the combined system will go into resonant vibration.

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* See Appendix A.

The harmonic torques of a reciprocating engine have frequencies proportional to the *engine speed*, hence such an absorber arranged for torsional oscillations could only be used to destroy the effect of *a given harmonic at one definite engine speed.*

A modification of this absorber, which successfully overcomes these defects, has been fitted to aircraft and other high speed engines and is called the *Pendulum Absorber.*

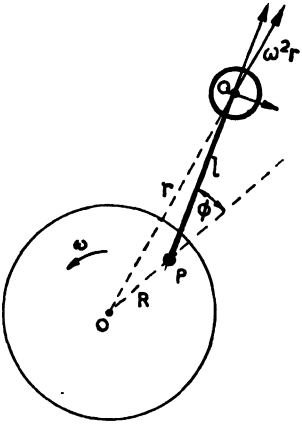

FIG. 87.

In the absorber considered above, the restoring force of the auxiliary system was obtained by means of the elastic coupling F_2 (*or* T_{02} in torsional equivalent), but in this new device the auxiliary vibrating system is essentially a simple pendulum *which is subjected to so large a centrifugal field that the gravitational field can be neglected.*

Fig. 87 shows a simple pendulum PQ attached at the point P to a disk which is revolving with uniform angular velocity ω about the centre O.

The pendulum is shown deflected through an angle ϕ from its mid position (the radial line through OP).

Let $PQ = l$, $OP = R$, and the instantaneous value of $OQ = r$, so that the pendulum is subjected to the centrifugal acceleration $\omega^2 r$.

The component of this acceleration perpendicular to PQ

$$= \omega^2 r \sin OQP.$$

Now
$$\frac{r}{\sin(180^\circ - \phi)} = \frac{R}{\sin OQP};$$

$$\therefore \text{ component} = \omega^2 R \sin\phi = -l\frac{d^2\phi}{dt^2}$$

(the negative sign appearing, as usual, because the acceleration is in the opposite direction to the displacement), and *provided* ϕ *is small* this becomes

$$\omega^2 R \, . \, \phi = -l \frac{d^2\phi}{dt^2},$$

or

$$\frac{d^2\phi}{dt^2} = -\omega^2 \frac{R}{l} \cdot \phi = -n^2\phi.$$

The motion of the pendulum is thus simple harmonic of frequency $\frac{n}{2\pi} = \frac{\omega}{2\pi}\sqrt{\frac{R}{l}}$ vibrations per sec.

$$= \text{revolutions per sec.} \times \sqrt{\frac{R}{l}} \cdot$$

The frequency of a simple pendulum under *gravity* is $\frac{1}{2\pi}\sqrt{\frac{g}{l}}$ so that, in subjecting the system to a *centrifugal* field only, we have replaced g by the centrifugal acceleration $\omega^2 R$, *the frequency now being proportional to engine speed* ω.

The absorber can thus be designed to eliminate the effect of any given harmonic to which the disk is subjected at all engine speeds.

It will be remembered that the order number of a harmonic

$$= \frac{\text{frequency}}{\text{revs. of engine}};$$

$$\therefore \text{ order number} = \sqrt{\frac{R}{l}} \cdot$$

Suppose an objectionable harmonic was of order 4,

then

$$4 = \sqrt{\frac{R}{l}} \cdot$$

The place usually chosen for attaching the pendulum is a crank web balance mass, and as space is obviously limited, the value of R can only be a few inches, say 4 in.

Thus, for operation against the above harmonic,

$$l = \tfrac{4}{16} = \tfrac{1}{4} \text{ in.}$$

This is so small that practical difficulties of design at once become apparent.

Another difficulty arises from the need for the pendulum to have as large a mass as possible, although this in no way affects the frequency.

Since the whole of the energy of the vibration has to be absorbed by the pendulum, a large mass is necessary in order to limit the amplitude, since our theory depends on ϕ being small.

Two ingenious devices to overcome these difficulties have been successfully employed, and both give the equivalent of a pendulum of very short length. The first consists of a cylinder free to roll in a hole drilled in the balance mass, but the requirement for a large mass cannot be fully met ; and in the second, the balance mass itself is suspended on two rollers.

For a complete description, and also for the theory of the absorber when damping is introduced, the reader is referred to more advanced works.

Another interesting type of absorber is gyroscopic in action and is described by Professor R. N. Arnold, D.Sc., in his paper " The Tuned and Damped Gyrostatic Vibration Absorber ".*

Elimination of the critical speeds of torsional oscillation of the crankshaft of multi-cylinder engines. As there are so many widely varying types of multi-cylinder engine installations, the methods to be adopted can only be determined by an examination of each case on its own merits. It will obviously be easier to ensure that the running speed of a constant speed engine for a land installation should not approach a critical than for a marine engine which has to operate over a range of speed.

The usual arrangement of Diesel installation comprises engine and flywheel direct coupled to a rotor, or, in the case of a marine engine, a propeller. Considering the latter, the shaft system therefore consists of engine masses, flywheel and propeller, and being direct drive, the engine will be of a slow speed type. If, as is usual, the propeller mass is small compared with that of the flywheel and engine masses, and the connecting shaft is long, then the node for the fundamental 1-noded mode of vibration will be situated near the engine and well removed from the propeller, which will therefore suffer the greatest amplitude during vibration. The shaft, however, being long, will have sufficient elasticity to withstand any amplitude likely to be reached by the propeller, which as long as it is immersed will be most efficiently damped.

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* *The Institution of Mechanical Engineers Proceedings* 1947, War Emergency Issue No. 25.

This system probably gives the least trouble from vibration, the 1-noded major being the only critical likely to cause any trouble. As the connecting shaft is made shorter and therefore stiffer for the same diameter, the presence of criticals within the operating range becomes more a source of anxiety, since the amplitude at which overstraining or fracture of the shaft will occur will be correspondingly less.

Continuing this idea, it will be apparent that if we replace the propeller by a rotor or other similar load, and thus have now no assistance from damping in keeping the amplitude down, the trouble will be intensified. All majors and minors of both the 1-noded and 2-noded modes may be important.

The most usual method to adopt for eliminating a critical is to alter the natural frequency of the shaft system so as to make the corresponding critical under observation well above or below the running speed.

This can be done by altering the magnitude of the attached masses or the stiffness of the shafting.

If the critical is brought below the running speed, it must be remembered that we have to pass through these resonant conditions as we bring the engine up to speed, with the consequent possibility of excessive amplitudes being forced, although the duration of these critical conditions will be short.

A change in the firing order will often reduce the magnitude of a minor critical, but at the same time may increase another that was negligible. The student should draw the vector diagrams for different firing orders and a given N.E.C.

Finally, there remains the addition of damping devices for altering the amplitude of a critical and also the speed at which resonance occurs. Dampers have been employed successfully for many years on petrol engines and a few of rather special design have been fitted to Diesels, but in general it is more usual to remove a critical from the operating range than to introduce damping.

The theory of such a damper is interesting and is developed for a simple case in a following section.

The problem of elimination of criticals becomes specially important with geared drives. We have seen that the natural frequencies of torsional oscillation of such systems can be determined provided that the teeth are always bedded firmly together by the transmitted torque, which must therefore never be allowed to become negative, otherwise the teeth will separate and then come together with violent impact on the torque reversing its sign again.

This chattering or hammering in the gear-box was of common occurrence on motor cars in their early days, but was not of sufficient magnitude to warrant investigation of any special nature, but on the introduction of the steam turbine for marine propulsion and the consequent necessity for a reliable reduction gear of massive design the trouble became acute. Not only did this hammering of the gears become so alarming that it was necessary to run the plant well below its designed speed, but in innumerable instances actual fracture of the teeth took place.

That the systems must have been in a state of forced torsional oscillation was considered to be the main cause of the trouble, although inaccuracies in the cutting of the teeth also intensified the matter.

Since any critical produces increased torque fluctuations in the shaft, in order to ensure that the torque at the gear face shall suffer no reversal in sign all criticals must be avoided.

The geared drive has recently been introduced for marine Diesels, and the vibrational problems become acute owing to the very irregular engine torque that is in any case transmitted.

In a steam turbine the torque transmitted is uniform, but disturbances tending to cause vibration are always present, as for instance the propeller blades revolving in a varying wake, etc.

The first attempt at a mathematical solution of the problem of elimination of this hammering action was made by Dr. J. H. Smith in his paper " Nodal Arrangements of Geared Drives " * in which he suggests that the shaft systems coupled to the gear-box should each be tuned to the same frequency, the gear-box thus acting as the node.

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* *Trans. Inst. Naval Architects* 1922, also *Engineering*, 1922.

An interesting account of the trouble experienced with a Diesel geared drive is given in the *American Motor Ship* (October 1927) in an article on the 10,000-ton motor-ship *Herman Falk.*

This ship was fitted with Falk geared engines, two 8-cylinder, 18 in. bore × 22·5 in. stroke, 4-stroke engines running at 275 r.p.m., being originally connected as shown in Fig. 88 (*a*) to the propeller shaft.

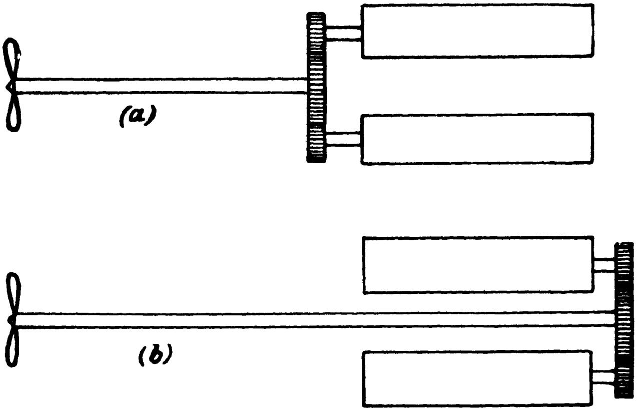

FIG. 88.

The gear wheel had 153 teeth, the pinions 50 teeth, thus the propeller speed was 90 r.p.m. With this arrangement trouble was experienced with criticals and the installation was altered to that shown in (*b*), thus increasing the elasticity of the shaft and lowering the 1-noded criticals away from the running speed.

With this new arrangement, the frequency of the 2-noded natural mode was calculated at 1,416 per minute.

The 8th order major would therefore occur at $1{,}416/8 \times 50/153 = 58$ r.p.m. of propeller, and was actually found to take place at 57 r.p.m.

The $5\frac{1}{2}$ order would appear at 83 r.p.m., and since this was very near the running speed of 90 r.p.m. it was found necessary to change the firing order to reduce the amplitude.

For the 3-noded mode the frequency was 2,780 per minute, and the 8th order was therefore 114 r.p.m. actually being found to occur at 100 r.p.m. This again was too near 90 r.p.m. and the critical was raised to 114 r.p.m. by increasing the stiffness of the coupling by 30 per cent.

(4) **The theory of the friction damper.** In the previous sections the use of a damper to reduce torsional oscillations has been mentioned, its use being most common in petrol engine practice. The device usually consists of a flywheel coupled to the shaft only through friction disks, or other frictional coupling. This flywheel tends to run at constant speed but will slip if the amplitude of vibration becomes excessive. A frictional retardation is thus introduced and at the same time, since the effective inertia of the system has been reduced, a change in the natural frequency occurs which destroys the resonant conditions.

Valuable ideas may be gained from a study of the behaviour of such a damper, and the differential equations for the simplest case as represented in Fig. 89 will now be developed, on the assumption that viscous friction, varying as the velocity, exists.

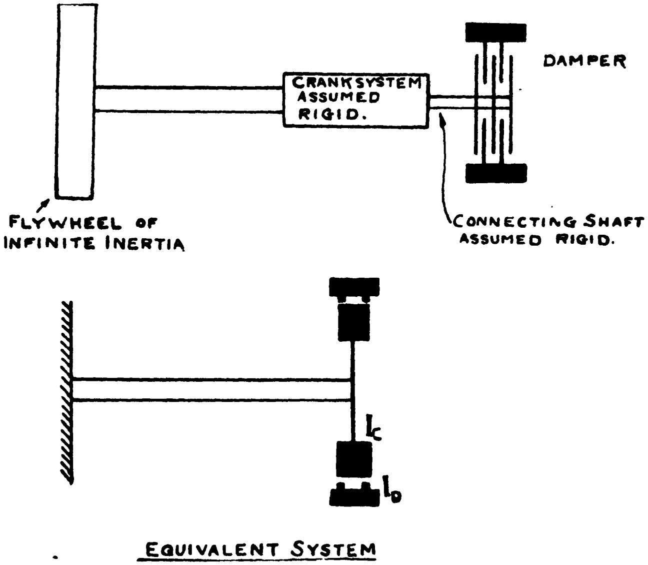
DAMPER
CRANKSYSTEM ASSUMED RIGID.
FLYWHEEL OF INFINITE INERTIA
CONNECTING SHAFT ASSUMED RIGID.
I_c
I_D
EQUIVALENT SYSTEM

FIG. 89.

We shall consider then a single load that can undergo angular vibration at the end of a shaft which is fixed at the other end. A simple harmonic disturbing torque acts on the load and a damper is frictionally coupled thereto.

This would represent a crankshaft system in which there was a heavy flywheel at one end whose moment of inertia compared with that of the crankshaft masses is so large that it may be assumed infinite, i.e. as far as the vibrations are concerned the node is at the flywheel.

The crankshaft itself is assumed to be rigid, i.e. of no elasticity, and all the crank masses are assumed to be concentrated at one point while the damper is coupled on by means of a rigid coupling.*

Let I_c = the moment of inertia of the load (crank masses);
I_a = the moment of inertia of the damper flywheel;
$T \cos qt$ = the periodic torque acting on the load;
T_0 = stiffness of equivalent shaft.

Since all vibrations will be superimposed on the steady rotation of the system under the action of a constant torque equal to the mean value of the actual torque, the displacement of the load at any instant will be measured from the position which the load would occupy under this mean torque; or alternatively, neglecting the actual rotation of the system the displacements will be measured, as usual, from the rest position of the load.

Hence, let

θ = the angular displacement of the load I_c at any instant from its mean position;

ϕ = the angular displacement of the damper flywheel at the same instant measured from the same datum.

Then the slip of the load relative to the damper flywheel is $\theta - \phi$ and the relative angular velocity of slip will be

$$\frac{d\theta}{dt} - \frac{d\phi}{dt} \quad \text{or} \quad \dot{\theta} - \dot{\phi}.$$

We shall assume that the friction varies as this relative angular velcoity, i.e. if T_1 is the friction torque per unit angular velocity of slip, the frictional resisting torque at any instant will be

$$T_1(\dot{\theta} - \dot{\phi}).$$

---

* A very complete investigation of this subject has been made by Major B. C. Carter in "Reports and Memoranda" of the Aeronautical Research Committee No. 1053 (E22), in which the effect of a finite flywheel and elasticity of the crankshaft and damper coupling are also fully discussed.

The Lanchester damper is of this type, since the friction plates are enclosed in a casing containing oil, so as to prevent the possibility of the plates seizing which might occur in time with dry contact.

The " laws of friction " for lubricated surfaces may therefore be assumed to apply—friction being proportional to the velocity.

The equations of motion can now be written down :

(1) *Load*

$$\underset{\substack{\text{Torque}\\ \text{accelerating}\\ \text{load}}}{\frac{I_c}{g}\ddot{\theta}} = \underset{\substack{\text{External}\\ \text{disturbing}\\ \text{torque}}}{T\cos qt} - \underset{\substack{\text{Friction}\\ \text{torque}}}{T_1(\dot{\theta}-\dot{\phi})} - \underset{\substack{\text{Restoring}\\ \text{elastic}\\ \text{torque}}}{T_0\theta}$$

or

$$\underset{\substack{\text{Torque}\\ \text{accelerating}\\ \text{load}}}{\frac{I_c}{g}\ddot{\theta}} + \underset{\substack{\text{Torque to}\\ \text{overcome}\\ \text{friction}}}{T_1(\dot{\theta}-\dot{\phi})} + \underset{\substack{\text{Torque to}\\ \text{overcome}\\ \text{elastic}\\ \text{constraints}}}{T_0\theta} = \underset{\substack{\text{External}\\ \text{torque}\\ \text{applied}}}{T\cos qt.}$$

Transferring now to the damper flywheel, the relative slip of damper to load is $\phi-\theta$, and the equation of motion is :

(2) *Damper*

$$\underset{\substack{\text{Torque}\\ \text{accelerating}\\ \text{flywheel}}}{\frac{I_d}{g}\ddot{\phi}} = \underset{\substack{\text{Frictional}\\ \text{torque}}}{-T_1(\dot{\phi}-\dot{\theta}),}$$

or

$$\frac{I_d}{g}\ddot{\phi} - T_1(\dot{\theta}-\dot{\phi})=0.$$

Solution, however, is simpler if we convert these equations into their vector form (see page 187) ; thus, representing the disturbing torque as $T^{i}e^{qt}$, let

$$\theta=Ae^{iqt},$$

$$\phi=Be^{iqt},$$

so that, substituting above, we get

$$(a)\quad -\frac{I}{g}q^2A+iT_1q(A-B)+T_0A=T,$$

$$(b)\quad -\frac{I_d}{g}\,q^2B + iT_1q(B-A) = 0,$$

or

$$\text{(i)}\quad -q^2A + i2kq(A-B) + n^2A = f,$$

$$\text{(ii)}\quad -qB + i\,.\,\frac{T_1g}{I_d}\,.\,(B-A) = 0.$$

where, as before, $f = \dfrac{Tg}{I_c}$, $n^2 = \dfrac{T_0g}{I_c}$, $2k = \dfrac{T_1g}{I_c}$.

Let also $\dfrac{I_d}{I_c} = R,$

$$\frac{q}{n} = \lambda,$$

$$\frac{2k}{n} = K \text{ (as before).}$$

Then $\dfrac{T_1g}{I_dq} = \dfrac{2k}{Rq} = \dfrac{K}{R\lambda} = M$ (say).

Thus dividing equation (i) by n^2, and equation (ii) by q, and making these further substitutions we get

$$(1)\quad -\lambda^2A + iK\lambda(A-B) + A = \frac{f}{n^2},$$

$$(2)\quad -B + iM(B-A) = 0.$$

Now from (2)

$$B = A\,.\,\frac{iM}{(iM-1)} \times \frac{(iM+1)}{(iM+1)} = A\,.\,\frac{M(M-i)}{1+M^2}.$$

and substituting this for B in (1) we get

$$A\left[-\lambda^2 + iK\lambda - \frac{iK\lambda M(M-1)}{1+M^2} + 1\right] = \frac{f}{n^2},$$

or $$A\left[1 - \lambda^2 + iK\lambda\left(1 - \frac{M^2}{1+M^2}\right) - \frac{K\lambda M}{1+M^2}\right] = \frac{f}{n^2};$$

$$\therefore\ A[\{(1-\lambda^2)(1+M^2) - K\lambda M\} + iK\lambda] = \frac{f}{n^2}(1+M^2),$$

or $$A[a+ib] = \frac{f}{n^2}(1+M^2);$$

$$\therefore\; A=\frac{f}{n^2}\cdot\frac{(1+M^2)}{(a+ib)}\cdot\frac{(a-ib)}{(a-ib)}$$

$$=\frac{f}{n^2}\cdot\frac{(1+M^2)}{a^2+b^2}[a-ib].$$

Thus, the magnitude of the vector represented by A, i.e. the amplitude of the load I_c, is

$$\theta_1=\frac{f}{n^2}\cdot\frac{(1+M^2)}{(a^2+b^2)}\cdot\sqrt{a^2+b^2}$$

$$=\frac{f}{n^2}\cdot\frac{(1+M^2)}{\sqrt{a^2+b^2}},$$

this vector lagging behind the disturbing torque vector by an angle whose tangent is $-\frac{b}{a}$, where

$$a=[(1-\lambda^2)(1+M^2)-K\lambda M],$$

$$b=K\lambda.$$

Now $B=\dfrac{A\,.\,M(M-i)}{(1+M^2)}$ and $A=\dfrac{f}{n^2}\dfrac{(1+M^2)}{(a^2+b^2)}[a-ib]$;

$$\therefore\; B=\frac{f}{n^2}\cdot\frac{M}{(a^2+b^2)}\cdot[(aM-b)-i(a+bM)].$$

Thus the magnitude of the vector represented by B, i.e. the amplitude of the damper flywheel I_d, is

$$\phi_1=\frac{f}{n^2}\cdot\frac{M}{(a^2+b^2)}\cdot\sqrt{(aM-b)^2+(a+bM)^2}$$

$$=\frac{f}{n^2}\cdot\frac{M}{(a^2+b^2)}\cdot\sqrt{(a^2+b^2)(1+M^2)}$$

$$=\frac{f}{n^2}\cdot\frac{M\sqrt{1+M^2}}{\sqrt{a^2+b^2}},$$

and lags behind the disturbing torque vector by an angle of tangent $-\dfrac{(a+bM)}{(aM-b)}$.

Now $\sqrt{a^2+b^2}$

$$\begin{aligned}
&=\sqrt{(1-\lambda^2)^2(1+M^2)^2+(K\lambda M)^2-2K\lambda M(1-\lambda^2)(1+M^2)+K\lambda^2}\\
&=\sqrt{(1-\lambda^2)^2(1+M^2)^2+K\lambda^2(1+M^2)-2K\lambda M(1-\lambda^2)(1+M^2)}\\
&=\sqrt{1+M^2}\,.\,\sqrt{(1-\lambda^2)^2(1+M^2)+(K\lambda)^2-2K\lambda M(1-\lambda^2)}\\
&=\sqrt{1+M^2}\,.\,\sqrt{(1-\lambda^2)^2+M^2+M^2\lambda^4-2\lambda^2M^2+K^2\lambda^2-2K\lambda M+2K\lambda M\lambda^2},
\end{aligned}$$

and, substituting $\dfrac{K}{R\lambda}$ for M, we get

$$\begin{aligned}
\sqrt{a^2+b^2}&=\sqrt{1+\left(\frac{K}{R\lambda}\right)^2}\\
&\quad\times\sqrt{(1-\lambda^2)^2+K^2\left[\frac{1}{R^2\lambda^2}+\lambda^2\left(\frac{1}{R^2}+1+\frac{2}{R}\right)-\frac{2}{R}\left(\frac{1}{R}+1\right)\right]}\\
&=\sqrt{1+\left(\frac{K}{R\lambda}\right)^2}\\
&\quad\times\sqrt{(1-\lambda^2)^2+K^2\left[\frac{1}{R^2\lambda^2}+\lambda^2\left(\frac{1}{R}+1\right)^2-\frac{2}{R}\left(\frac{1}{R}+1\right)\right]}\\
&=\sqrt{1+\left(\frac{K}{R\lambda}\right)^2}\cdot\sqrt{(1-\lambda^2)^2+K^2\left[\left(1+\frac{1}{R}\right)\lambda-\frac{1}{R\lambda}\right]^2}.
\end{aligned}$$

Thus finally,

$$\theta_1=\frac{f}{n^2}\cdot\frac{\sqrt{1+\left(\dfrac{K}{R\lambda}\right)^2}}{\sqrt{(1-\lambda^2)^2+K^2\left[\left(1+\dfrac{1}{R}\right)\lambda-\dfrac{1}{R\lambda}\right]^2}}$$

and

$$\phi_1=\frac{f}{n^2}\cdot\frac{\dfrac{K}{R\lambda}}{\sqrt{(1-\lambda^2)^2+K^2\left[\left(1+\dfrac{1}{R}\right)\lambda-\dfrac{1}{R\lambda}\right]^2}}.$$

Now our chief concern is to discover how the *magnification factor* is affected by the different variables present. This factor, it will be remembered, is given by the ratio

$$\begin{aligned}
\frac{\text{Actual amplitude of load}}{\text{Amplitude under static torque } T}&=\frac{\theta_1}{T/T_0}=\frac{\theta_1 T_0}{T}\\
&=\frac{\text{Maximum shaft torque}}{\text{Maximum disturbing torque}}=\frac{\theta_1}{f/n^2},
\end{aligned}$$

and thus

$$\text{M.F.} = \sqrt{\frac{1+\left(\dfrac{K}{R\lambda}\right)^2}{(1-\lambda^2)^2+K^2\left[\left(1+\dfrac{1}{R}\right)\lambda-\dfrac{1}{R\lambda}\right]^2}}.$$

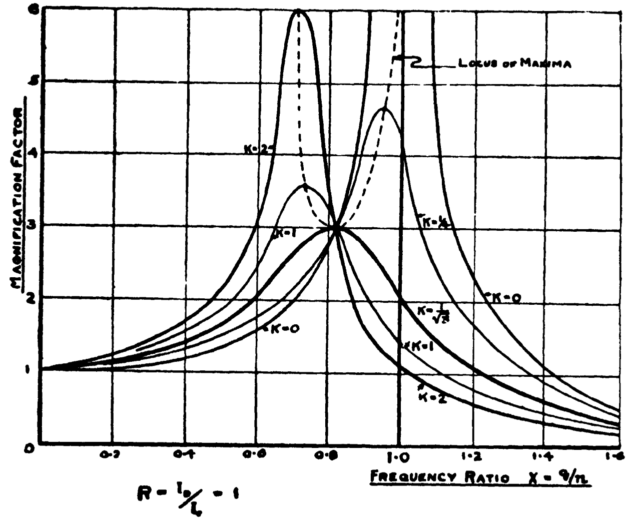

FIG. 90.

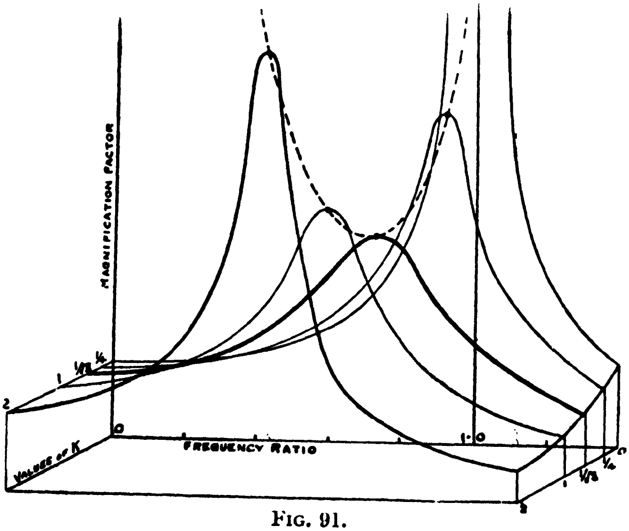

FIG. 91.

As in our previous discussions, for any given system K and R will be constants, *so that the magnification factor depends only on the frequency ratio* $\lambda = q/n$.

If therefore, *for any given value of* R, we plot the M.F. against the frequency ratio λ, we shall obtain a curve for each value of K chosen, as shown in Fig. 90, where the curves have been plotted for the value $R=1$.

It will be noticed that when $K=0$, the M.F. becomes $\frac{1}{1-\lambda^2}$, i.e. $\frac{n^2}{n^2-q^2}$ as before.

As damping increases, the maximum value of the M.F. occurs *before* coincidence of the forced and natural frequencies, diminishing at first *and then increasing* again.

Also it will be noticed that each curve passes through a common point which is the minimum value of the maxima of the various curves.

Thus, if the running speed is such that the frequency ratio is above that corresponding to this common point, fitting a damper of this type will *reduce* the amplitude, since the curves for damping lie below the curve for $K=0$; but, if the frequency ratio is below the common point value, a damper will *increase* the amplitude and may cause serious oscillations, e.g. :

q/n	K	$M.F.$
1·2	0	2·3 approx.
1·2	2	0·6 ,,
0·7	0	2·0 ,,
0·7	2	6·0 ,,

To find the point of minimum maximum :

(*a*) Differentiate with respect to λ, K being constant.

(*b*) Differentiate with respect to K, λ being constant.

These two expressions must simultaneously equal zero, i.e. the first equation then always gives maximum M.F. for any value of K, or in other words it is the equation of the dotted curve (Fig. 90), whilst the second expression being simultaneously zero with the above gives the minimum value of this curve.

Fig. 91 shows the same curves in solid projection. Thus,

(*a*) $d\theta_1/d\lambda=0$.

Here differentiating with regard to λ^2, we get:

$$\text{(i)}\ \lambda_m^6+\lambda_m^4\left[K^2\left(\frac{2}{R^2}+\frac{1}{R}+\frac{1}{2}\right)-1\right]+\frac{K^2}{R^2}\lambda_m^2\left[K^2\left(1+\frac{1}{R^2}\right)^2-2\right]-\frac{K^4}{R^3}\left(1+\frac{1}{R}\right)=0,$$

where λ_m is the value of the frequency ratio at which the maximum value of the M.F. occurs for any chosen value of K.

(*b*) $d\theta_1/dK=0$.

Performing this differentiation, we shall obtain

$$\text{(ii)}\qquad \underset{\text{(Minimum)}}{\lambda_m}=\sqrt{\frac{2}{R+2}},$$

a result which does not contain K, which means that with this value of λ the M.F. is the same for any value of K, as is shown by the curves.

Substituting this value λ_m in the expression for the M.F. we get the minimum maximum value:

$$\text{M.F.}=\left(1+\frac{2}{R}\right),$$

and substituting this same value λ_m in equation (i) above we get the corresponding value of K, which makes λ_m the minimum maximum.

This results in

$$K_m=R\sqrt{\frac{2}{(R+1)(R+2)}}.$$

From these results in any given case, we can determine the engine speed at which the amplitude becomes a mimimum however the damper is adjusted, i.e. whatever the value of K.

For the value $R=1$ for which the graphs have been drawn

$$\lambda_m=\sqrt{\frac{2}{3}},$$

$$K_m=\frac{1}{\sqrt{3}},$$

and the minimum maximum M.F. $=3$ as will be observed.

Note that if $R = \infty$, i.e. the damper flywheel has infinite inertia, and suffers therefore no movement but exerts only a damping effect, the expression for the M.F. reduces to that on page 159 for a forced damped vibration as we should expect. The curves resulting are similar to those of Fig. 53*a*, the common point occurring when $\lambda = 0$.

CHAPTER XI

THE EFFECT OF DYNAMIC LOADS ON AN ELASTIC SYSTEM. WAVES

On page 49 we noted that the stress produced in an elastic rod due to the application of a " live load " was twice that which would be produced if the same load was applied statically. Commander C. W. Chapman * has made a most interesting examination of the effects of suddenly applied loads on an engine system.

Suppose, as a simple example, we take the case of an elastic support carrying a single load W which is subjected to a disturbing force varying directly with time, i.e. $P=Rt$. This could be imagined as the piston and connecting rod assembly of an engine subjected to the explosion impulse.

The equation of motion will be

$$\frac{W}{g}\frac{d^2x}{dt^2}=-Fx+Rt,$$

or

$$\frac{d^2x}{dt^2}+n^2x=\frac{R\cdot g}{W}\cdot t=f\cdot t.$$

Now, under *static* application of the force Rt, at a time t the displacement of the load would be $\frac{Rt}{F}$ or $\frac{ft}{n^2}$ which is a solution of the above equation ; but to this we may add the motion resulting when the right-hand side of the equation is zero, i.e. $A\cos nt+B\sin nt$, being the free vibrations existing independently of the forced.

The whole solution is therefore

$$x=\frac{Rt}{F}+A\cos nt+B\sin nt.$$

---

* " Dynamic Loading and some indications of its effect on Internal Combustion Engines " by Commander C. W. Chapman, R.N.V.R., M.Eng., M.I.Mech.E. *Proceedings Inst. Mech. Engineers*, Vol. 153, 1946.

To evaluate this we must know the displacement and velocity of the free motion at the instant that the impulse is applied.

Let us assume, as in previous work, that when the force is applied the load is at rest, with no initial displacement, i.e. when $t=0$:

$$x_0=0\,; \quad \therefore\ A=0\,;$$

$$\frac{dx_0}{dt}=0=\frac{R}{F}+Bn\,; \quad \therefore\ B=-\frac{R}{Fn}.$$

$$\therefore\ x=\frac{R}{F}t-\frac{R}{F}\cdot\frac{1}{n}\sin nt,$$

or the force exerted in the structure

$$=Fx=Rt-\frac{R}{n}\sin nt.$$

Thus, as we might expect, the displacement x is equal to the static (due to the applied force) upon which is superimposed the displacement due to a free vibration. This latter vibration starts in opposition to the displacement due to the applied force as is shown by the negative sign.

If $T=$ the natural period of vibration so that $n=\frac{2\pi}{T}$, we may write

$$Fx=Rt-R\,.\,\frac{T}{2\pi}\sin\left(\frac{2\pi}{T}\right)t,$$

so that the modification of the static force due to the free vibration depends on (1) R, the rate of loading and (2) T, the period of the free vibration, and, as time is therefore the important factor, Chapman prefers to substitute $\frac{2\pi}{T}$ for the circular frequency n.

Fig. 92 shows the static displacement curve $\frac{R}{F}\cdot t$ and the natural vibration separately and superimposed.

Suppose at a time t_1 the applied force ceased to increase and became constant (or was " stabilised ") at this value producing, therefore, a static displacement $\frac{R}{F}t_1=ab$ (Fig. 92), the total displacement at this instant will be

$$x=\frac{R}{F}t_1-\frac{R}{F}\cdot\frac{1}{n}\sin nt_1, \text{ i.e. } cb. \quad \dots\dots\dots\dots\dots\dots\text{(i)}$$

The load will now vibrate with its natural frequency about this constant static deflexion position and the amplitude of the motion will depend on the displacement of the load from this position and its velocity at the instant of stabilisation of the force.

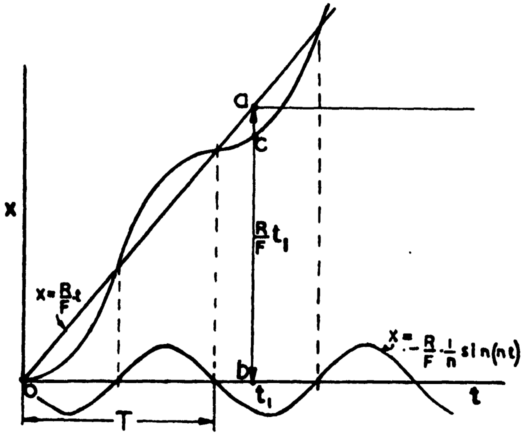

FIG. 92.

At any time $t > t_1$ or at a time $(t - t_1)$ measured from stabilisation, the motion about the static position will therefore be of the form

$$x = C \cos n(t - t_1) + D \sin n(t - t_1), \quad \text{..............(ii)}$$

or

$$= a \cos [n(t - t_1) + e].$$

At a time t_1, or when $(t - t_1) = 0$,

the initial displacement *from the static position* from equation (i)

$$= -\frac{R}{F} \cdot \frac{1}{n} \sin nt_1 \text{ (i.e. } ac\text{, Fig. 92)}$$

$$= C \text{ from equation (ii) ;}$$

and similarly from equations (i) and (ii),

$$\text{Initial velocity}' \frac{dx_0}{dt} = \frac{R}{F}\left(1 - \frac{1}{n} \,.\, n \cos nt_1\right) = nD,$$

or

$$D = \frac{R}{F} \cdot \frac{1}{n} (1 - \cos nt_1).$$

The free vibration about the static position $\frac{R}{F}t_1$ is given by substituting for C and D in equation (ii), and the amplitude of the motion is

$$a=\sqrt{C^2+D^2}$$

$$=\frac{1}{n}\cdot\frac{R}{F}\sqrt{2(1-\cos nt_1)}=\frac{1}{n}\frac{R}{F}\sqrt{2\left(2\sin^2\frac{nt_1}{2}\right)}$$

$$=2\cdot\frac{1}{n}\cdot\frac{R}{F}\sin\frac{nt_1}{2}.$$

It is useful to compare this amplitude with the static deflexion $\frac{R}{F}t_1$ produced by the force, i.e.

$$\frac{\text{Amplitude of free vibration}}{\text{Static deflexion due to force}}=\frac{a}{\frac{R}{F}t_1}=\frac{2}{nt_1}\sin\frac{nt_1}{2},$$

or, substituting for $n=\frac{2\pi}{T}$,

$$\underline{=\frac{T}{\pi t_1}\sin\frac{\pi t_1}{T}.}$$

Thus the amplitude reached depends on the time t_1 up to stabilisation of the force and the periodic time T of the free vibrations, t_1/T being the deciding factor for a given elastic system.

If t_1 is very small, i.e. a suddenly applied force, we may write $\frac{\pi t_1}{T}$ (radians) for $\sin\frac{\pi t_1}{T}$, hence the above ratio is approximately unity, i.e. the *total* displacement (and therefore stress) is twice that due to static application of the force, a result already noted.

The greater t_1, the less does the above ratio become and thus the less the total displacement.

On page 148 we examined the case for a similar system at rest initially and then subjected to a simple harmonic force $P\cos qt$. Since this force was given as a cosine function, when we started to measure time (i.e. $t=0$) the maximum value P was acting. If, however, at the start the force had been zero and had risen to its maximum value it would have been represented by $P \sin qt$, and the resulting motion would have been

$$x = \frac{f}{n^2 - q^2}\left(\sin qt - \frac{q}{n}\sin nt\right)^*$$

being again the superposition of a free motion on the forced.

In his paper Chapman considers the effect of stabilising such a force at a given time and again shows the importance of time on the resulting motion.

Waves. In all our work up to this point we have assumed that the effect of a disturbing force is transmitted through the elastic connexions instantaneously, but if these connexions are of considerable length the time element will again be important, since time will be required for the effect of an applied force to travel along the system in the form of a wave.

Sound, light, radiant heat, etc., are all transmitted by wave motion and the subject is fully dealt with in text-books of physics, but it will be of interest here to recall the elementary principles in order to indicate the importance of wave transmission in various engineering problems.

In the previous chapters we have considered the periodic motion of various elastic systems in which, assuming no damping, the component particles moved in phase or 180 degrees out of phase. Let us now examine the motion produced when the component particles of a medium are performing simple harmonic motions of the same period and amplitude, but the phases of which are different, yet at the same time are related to one another in a definite way.

Consider a number of particles arranged at equal distances along a straight line AB, and imagine them to be vibrating in a direction perpendicular to AB with the same period T, but each successive particle is (say) 45 degrees out of phase with the preceding one. Fig. 93*a* shows the relative positions when point (1) is at its maximum displacement, together with the respective vectors at 45°, which generate on the lines perpendicular to AB, the simple harmonic motions of the particles. Fig. 93*b* shows the positions after the vectors have rotated through 45°, i.e. after a time interval $T/8$ and (*c*) for a similar successive interval.*

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* See Examples III, Question 2.

From examination of these diagrams it will be clear that the particles at any instant are situated on a harmonic curve

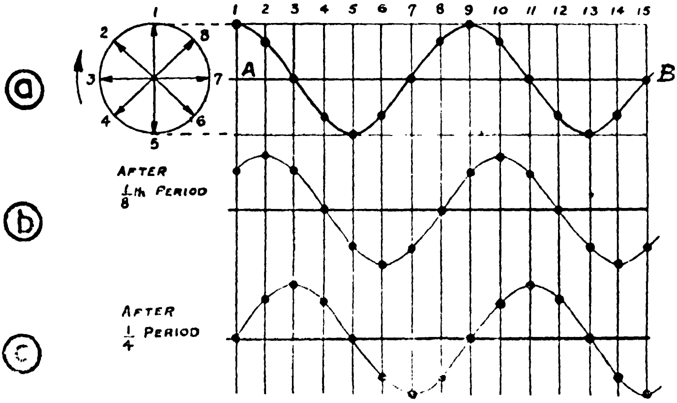


FIG. 93.

which *appears* to travel along the line AB as time proceeds. This curve is called a wave, and the distance moved in a complete period of the S.H.M. of the particles is called the wave-length, or alternatively it is the distance between two consecutive particles that are in the same phase.

Note that although the form of the wave is similar to that of the harmonic curve shown in Fig. 2, this latter is the *space-time* curve for a *single* particle, whilst in our present case the wave form gives the actual instantaneous positions of a series of particles.

It must further be emphasised that the translatory motion of the wave is not accompanied by the translation of the particles themselves, which merely perform their individual motions at right angles (in this case) to the path of the wave. A very familiar illustration of this is the wave motion produced along a rope by shaking one end in a vertical plane.

The wave motion just examined is caused by a transverse vibration of the particles and a visual wave results in many cases. Waves may, however, be transmitted through a medium by the longitudinal vibrations of the particles, i.e. by the particles executing their S.H.M's. in the direction in which the waves move. In this case, the particles no longer remain a fixed distance apart, but are alternately crowded together and drawn apart, but again there is no translation of the particles.

---

* The vectors in Figs. 93 and 94 are shown rotating clockwise.

Consider a long tube filled with air, water, or other elastic medium of continuous particles, and fitted with an elastic diaphragm at one end, which can be set into vibration in a plane perpendicular to the axis of the tube.

As the diaphragm moves inward the particles of the medium in contact with it are displaced along the tube, and, coming into contact with adjacent stationary particles, produce a zone of compression at the diaphragm. The first particles, due to their acquired velocity, must also impart motion to the neighbouring particles, which will in turn similarly affect succeeding particles so that a zone of compression travels forward along the tube.

On the return of the diaphragm outwards, the particles of the medium in contact with it will reverse their motion, causing a reduction in pressure as the individual particles move away from one another in the direction of motion of the diaphragm. This carries on throughout the whole length of the tube and thus a zone of low pressure is transmitted. Immediately afterwards on the forward movement of the diaphragm, another zone of high pressure is sent forward, and these alternating zones of high and low pressure will continue as long as the diaphragm performs its vibration.

All the particles in turn are thus impressed with the same S.H.M. as the diaphragm, but as it takes time to transmit the motion, there will be a phase difference between each succeeding particle in exactly the same manner as for a wave caused by a transverse vibration, and if we were to plot the increase and reduction in pressure above and below the mean pressure for successive points along the tube at any instant, a simple harmonic curve would result. This wave will travel forward along the tube and its crests correspond to high-pressure zones and its troughs to low.

The distance between two consecutive particles in phase is the wave-length as before. Fig. 94 shows a series of particles each vibrating with S.H.M. of the same amplitude and period in the line *AB*, but differing in phase by 45°. The positions shown are for time intervals of $\frac{1}{8}$ period, and the three particles nearest together are indicated to make clear the travel of the compression zone.

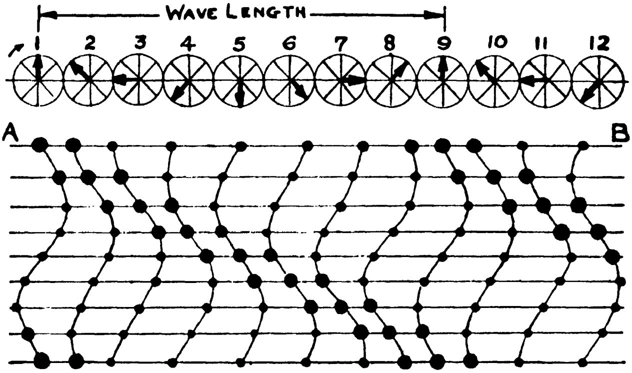


FIG. 94.

Reflexion of waves. When a zone of compression, travelling through a medium, say an elastic rod, encounters a rigid obstruction an exactly similar wave will be reflected back from the obstructing surface. We can visualise the effect as follows: as the compression zone moves forward each layer of particles in turn imparts a forward motion to the next stationary layer. Any layer could be converted into a rigid obstruction by moving it backwards by exactly the same amount as the forward motion imparted to it by the arrival of the compression wave. This backward motion starts off the reflected wave, and the particles in the vicinity, which are already under compressive stress due to the forward wave, will now be subjected to twice this stress due to the added effect of the return wave.

The same reasoning will apply when a zone of tensile stress is transmitted, except that now the actual movement of the particles will be in the opposite direction to the travel of the wave.

Suppose a zone of *compression* arrives at the *free* end of the elastic rod. Since the last layer of particles can suffer no stress it will therefore move through a greater distance than if subjected to the compressive stress. This extra movement puts the end layers into *tension* and a *tensile wave* is reflected back.

The tension and compression must neutralise the stress at the free end, so that the movement of the final layer under tension must equal that due to compression, or the movement of this layer will be twice that due to the one stress alone.

If two or more waves are passing a section at the same time their resultant effect will be the algebraic sum of their separate effects.

An alternative way of viewing the reflexion of a wave is as follows (Fig. 95):

If two exactly similar zones of compression (or tension) are transmitted along the rod from *opposite directions*, the layer of particles at which the wave fronts meet will be kept stationary and will remain so whilst the waves pass over one another, since it will be equidistant from both fronts and therefore subjected to the same stress from both waves. Fig. 95 shows two such zones, in which the ordinates represent the stress (or strain) transmitted which is reducing as the waves proceed. They are shown approaching and passing one another with equal velocities *c*. Considering either side of this stationary layer we see that a rigid obstruction is equivalent to the reflexion of the forward wave, the final layer being subjected to twice the stress due to the forward wave at any instant.

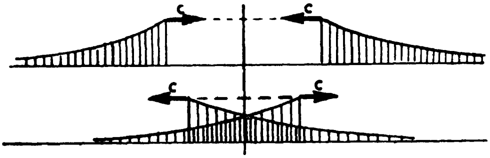


FIG. 95.

If, however, one of the waves is *tensile* and the other *compressive*, this central layer will be subjected to zero stress during their passage, and its movement will be twice that due to a single wave, since the movement of the particles in each of the opposing waves will be in the *same* direction. This is equivalent to the conditions at the free end of a rod, the reflected wave being of opposite sign to the forward wave.

Consider again the tube in which an alternating wave of high and low pressure zones is being transmitted (Fig. 96). Let the length of the tube be an exact number of wave-lengths and great compared with the amplitude of the particles in vibration ; and let the end opposite the diaphragm be closed by a rigid plug, so that when the wave reaches this point it will be reflected back. Since the length has been made an exact number of wave-lengths, the first zone of high pressure of the reflected wave will arrive at the diaphragm as it is about to send another along, so that now a zone of twice the original pressure will be sent forward.

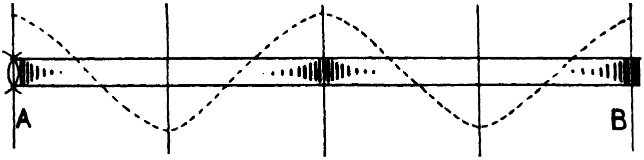


FIG. 96.

This in turn will be reflected back to the diaphragm and so waves of ever-increasing pressure, with corresponding greater amplitudes of oscillation of the transmitting particles, will be sent out until finally the pipe bursts.

Suppose now that instead of a diaphragm at *A* we substitute a small piston capable of being driven with S.H.M. by (say) a crank and long connecting rod, and we also place a similar mechanism at the other end *B*, which is a complete number of wave-lengths from *A*. Then it will be clear that the energy transmitted by the wave produced by the oscillation of the piston *A* is capable of driving *B* at the same speed as *A* provided *B* is set in phase with *A*, since, if the pipe were continued *past* *B*, the latter would produce in it a continuation of the original wave. If *B* was placed at any other point in the pipe-line, its phase of motion would differ from that of *A*, being such as to produce a continuation of the original wave beyond *B*. Hence, wherever *B* is placed, provided it can rotate at the same speed as *A*, it is capable of utilising the whole of the energy transmitted from *A*.

If, however, *B* is not able to take the whole of the energy transmitted by *A*, then the portion not absorbed will form a reflected wave which will, on arrival at *A*, increase the energy transmitted until, as with a closed pipe, bursting occurs. Thus some means of storing the surplus energy and utilising it at any desired time is necessary.

From this very brief outline it will be realised that it is quite possible to transmit power by wave motion, i.e. without actual transference of the particles of the medium itself. This method of transmission was first introduced by G. Constantinesco.

The effects of wave reflexion in increasing the pressure in a pipe-line is of great importance in many spheres of engineering—for instance, in a Diesel engine using a "solid injection" fuel system operated by a jerk pump, the oil pressures and the timing of injection are considerably influenced by waves set up in the pipe-line between the pump and the nozzle.

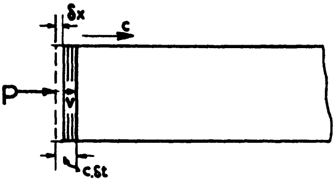


FIG. 97.

The velocity c of wave propagation in a pipe or a uniform elastic rod can be obtained quite simply. In Fig. 97, let

A = the cross-sectional area of the rod ;

P = the suddenly applied force at one end setting up a uniform stress $f=\frac{P}{A}$ on an elementary layer at the end of the rod.

Unless P varies the stress will be uniform and will be transmitted to each layer in turn, so that in a time δt a length of rod $=c\,.\,\delta t$ will have been affected, where c = the velocity of transmission of the wave.

Due to this compressive stress the length $c\,.\,\delta t$ will have shortened by the amount δx, where

$$\text{stress} = E \times \text{strain},$$

or
$$f = E \times \frac{\delta x}{c\,.\,\delta t},$$

i.e.
$$\delta x = \frac{f}{E} \times c\,.\,\delta t.$$

Then the velocity v *of the particles* in this elementary zone of compression (of length $c\,.\,\delta t$) is dx/dt, δx being the distance moved in the time δt, so that

$$v = \frac{f}{E} \times c.$$

Again, the length $c \, . \, \delta t$ of the rod has acquired momentum in the time δt, the velocity of its particles having been changed from zero to v, so that the change of momentum is

$$A(c \, . \, \delta t)\rho \, . \, v$$

where $$\rho = \frac{\text{wt. per unit volume}}{g}.$$

Thus the *rate* of change of momentum will be

$$Ac\rho v$$

which must be equal to the force applied,

i.e. $$P = fA = Ac\rho v = Ac\rho\left(\frac{f}{E} \cdot c\right),$$

or $$c = \sqrt{\frac{E}{\rho}},$$

and *is entirely independent of the stress transmitted.* This is, in other words, the velocity of sound in the medium concerned.

In certain research work it may be required to subject a specimen to a heavy stress for varying, but very small, intervals of time. This may be done by allowing a weight to fall upon a stop at the lower end of a long vertical wire, thus transmitting a wave of tensile stress along the wire to the specimen at the top. The period during which the weight is in contact with the stop can be varied, thus varying the length of the tensile stress zone transmitted.

Wave transmitted along an elastic wire by a falling weight.

Let m = the mass of the falling weight,

m_s = the mass of the stop (if not negligible),

h = the height from which the mass is dropped,

then the velocity of the weight just before impact $= \sqrt{2gh}$, and by equating momentum before and after, the velocity V of the weight and stop just after impact $= \frac{m}{m + m_s}\sqrt{2gh}$. Now in a very short time δt after impact, during which the velocity V has not appreciably altered, the displacement of the stop $= V \, . \, \delta t$, and the distance travelled by the wave is $c \, . \, \delta t$.

Thus the mean strain in the stressed zone $=\frac{V\,.\,\delta t}{c\,.\,\delta t}=\frac{V}{c}$, a result noted above in a slightly different way.

The wave travels up the wire to the specimen or fixed end and is there reflected back, producing momentarily at this point a strain $\frac{2V}{c}$ or a corresponding stress $=E\times\frac{2V}{c}$ as before described.

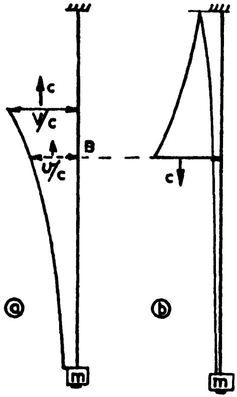


FIG. 98.

The wave or zone of tension continues to be transmitted as long as motion of the weight after impact occurs. This motion however, is being retarded so that the strain and corresponding stress, which are proportional to the velocity of the weight at the instant of transmission, are continually diminishing as one moves back from the wave front. This is shown by the curve of strain in Fig. 98*a* which represents the amplitude or strain at each section of the wire as the wave moves forward.

The initial strain, i.e. the strain produced at the wave front, is, as above, $\frac{V}{c}$.

Let $v=$ the velocity of the weight and stop at any instant after impact, so that the strain which will be transmitted at this instant is $\frac{v}{c}$, and therefore the retarding *force* exerted by the wire on the weight is $E\,.\,\frac{v}{c}\times A$, where A is the area of cross-section of the wire.

The equation of motion of the weight is therefore

$$\frac{m}{g}\frac{d^2x}{dt^2}=-\frac{v}{c}\,.\,EA,\quad \text{where } v=\frac{dx}{dt};$$

$$\therefore\ \frac{m}{g}\frac{dv}{dt}=-\frac{v}{c}\,.\,EA,$$

or

$$\frac{dv}{v}=-\frac{EAg}{c\,m}\,.\,dt,$$

and integrating,

$$\log_e v=-\frac{EAg}{c\,m}+\text{constant}.$$

When $t=0$, $v=V$;

$$\therefore \text{ the constant} = \log_e V,$$

so that
$$\log_e \frac{v}{V} = -\frac{EAg}{cm}\cdot t$$
$$= -\frac{cA\rho g}{m}\cdot t,$$
$$= -\frac{c\mu}{m}\cdot t,$$

where $A\rho g = \mu =$ mass per unit length of the wire,

or
$$\underline{v = V \cdot e^{-\frac{c\mu}{m}\cdot t}},$$

t being measured from the start of impact.

The strain at any point of the wire is zero until the wave reaches it when it becomes $\frac{V}{c}$, and then steadily reduces to $\frac{v}{c} = \frac{V}{c}\cdot e^{-\frac{c\mu}{m}\cdot t}$, continuing until the reflected wave front reaches this point when the strain will be *increased by* $\frac{V}{c}$, the sum of the two being the maximum strain to which this section will be subjected (Fig. 98*b*).

CHAPTER XII

TRANSVERSE VIBRATION OF BEAMS

A LOADED beam is an elastic structure and may therefore be set into free natural vibration.

By " transverse " vibration is meant a vibration in any given plane of bending, the system therefore having the same number of degrees of freedom as the loads carried, since the position of each load can be stated in terms of a single coordinate. In many cases, such as a shaft, the system is free to vibrate in any plane and is not constrained to a particular one, so that its motion will be the resultant of the respective motions of each vibration.

For instance, considering any one particular point in the beam, if this undergoes a S.H.M. $x_1 = y \cos(\omega t + e)$ in one plane and a similar motion $x_2 = y \sin(\omega t + e)$ in a plane at right angles, then we have seen from page 30 that the resultant motion is in a circle of radius y and with angular velocity ω. Thus as every point in the beam has similar motions, the whole beam will undergo a circular vibration with angular velocity ω or frequency $\omega/2\pi$. This conception is of extreme importance, and will be dealt with again in the next chapter.

In the following investigations we shall assume that all parts of the shaft move with S.H.M. along parallel straight lines. This will be approximately true provided that, as is usual, the amplitude of the motion is small compared with the span. Also we shall assume that the plane of any disk, etc., remains the same throughout the vibration. Actually, except for the special case of a single load at the centre of a uniform shaft which is supported at its ends, the plane will vary, the load being subjected to an angular oscillation about an axis perpendicular to the plane of the transverse vibration.

However, if the radius of gyration of the load is small compared with the span of the shaft, the effect of the " rotatory inertia " may be neglected. Its effect in any case is always to exert an opposing bending moment which will tend to stiffen the shaft against the vibration, so that the true frequency will be slightly higher than that found without taking it into account.

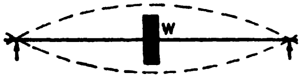


FIG. 99.

Consider first a beam or shaft of negligible mass carrying a single concentrated load W (Fig. 99). The equation of motion for the natural transverse vibration of this system in any given plane, say the vertical, will be as before:

$$\frac{W}{g}\frac{d^2y}{dt^2} = -Fy,$$

where y is the deflexion of the load at any instant from its rest position, and F is the stiffness of the beam at this point, i.e. the force to produce unit deflexion there.

The rest position with a horizontal shaft would, due to gravity, be the static deflexion position.

We have previously noted, when considering a load on a rod, etc., that gravity does not affect the vibration but only alters the rest position of the load when the rod is vertical from that taken up when the rod is horizontal.

Actually in the case of a beam gravity will affect matters slightly, since if the beam were vertical the vibration would be about the straight vertical rest position, but if horizontal, the rest position is the static deflexion curve, hence the configuration of the system is altered.

We shall, however, neglect gravity, or eliminate it by considering a vertical beam.

The above equation of motion represents, of course, S.H.M., the frequency being

$$\frac{1}{2\pi}\sqrt{\frac{Fg}{W}} = \frac{1}{2\pi}\sqrt{\frac{g}{Y_s}},$$

where $Y_s = \frac{W}{F}$ = the static deflexion of the load.

If there are two loads on a beam it will be clear that two modes of vibration are possible: (1) both loads moving together, or (2) the loads moving in opposite directions (Fig. 100).

Similarly with three loads, three modes of vibration are possible; and continuing, it will be seen that with an infinite number of loads, i.e. a distributed load, an infinite number of modes of vibration can occur. In each case the fundamental or slowest mode is that which has the least number of nodes, i.e. with a cantilever or a beam on two supports, when there is no node along the beam (except at the supports which form nodes, of course), the beam bending into a single bow, this being the shape of least constraint.

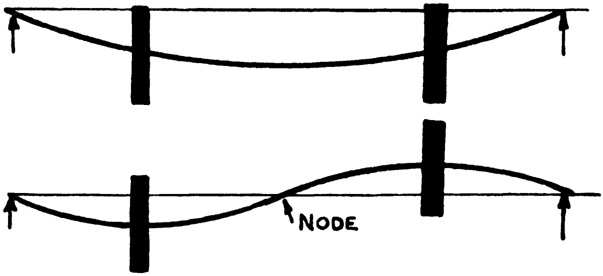


FIG. 100.

Lord Rayleigh, in his book *The Theory of Sound*, has made complete mathematical investigations into the subject, deducing general equations for the frequencies for any type of loading and support.

Let us consider a beam carrying a uniformly distributed load.

Transverse vibrations of a beam carrying a uniformly distributed load (Fig. 101).

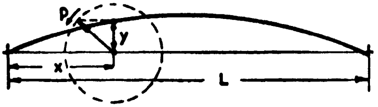


FIG. 101.

Let w = the weight per unit length;

L = the length of the beam;

y = the deflexion at a distance x from one support at any time t.

Then, assuming that each element of the beam vibrates with simple harmonic motion of frequency $p/2\pi$, we have

$$\frac{d^2y}{dt^2} = -p^2y. \quad \dots\dots\dots\dots\dots\dots\dots\dots (1)$$

Now in books on the "Strength of Materials" it is shown that the *loading per unit length* of beam causing a deflexion y of the element concerned is given by

$$EI\frac{d^4y}{dx^4} \quad \text{(see page 281)}$$

where I = the second moment of area of cross-section of the beam about the neutral axis.

When the deflexion is the static this loading, of course, is w, but, when the beam is vibrating, the force required to accelerate each element *of unit length* is

$$\frac{w}{g}\frac{d^2y}{dt^2},$$

and therefore the inertia reaction of each element on the beam is

$$-\frac{w}{g}\frac{d^2y}{dt^2}.$$

In other words this is the loading per unit length to which the beam is subjected at any given instant.

Thus $$-\frac{w}{g}\frac{d^2y}{dt^2}=EI\frac{d^4y}{dx^4},$$

and therefore from (1)

$$p^2y=\frac{gEI}{w}\cdot\frac{d^4y}{dx^4},$$

or $$\underline{\frac{d^4y}{dx^4}-m^4y=0}, \quad \text{where } m^4=\frac{wp^2}{gEI}.$$

This equation is true for any value of y, and is a standard differential equation, the solution of which is

$$\underline{y=A_1e^{mx}+B_1e^{-mx}+C\cos mx+D\sin mx,}$$

or, alternatively,

$$=(A_1+B_1)\cosh mx+(A_1-B_1)\sinh mx+C\cos mx+D\sin mx,$$

i.e. $$\underline{y=A\cosh mx+B\sinh mx+C\cos mx+D\sin mx.}$$

$$\left[\cosh mx=\frac{e^{mx}+e^{-mx}}{2}, \qquad \sinh mx=\frac{e^{mx}-e^{-mx}}{2};\right.$$

$$\therefore\ 2\cosh mx=e^{mx}+e^{-mx}, \qquad 2\sinh mx=e^{mx}-e^{-mx};$$

$\therefore$ Sum $\cosh mx+\sinh mx=e^{mx}$, ...(1)

Difference $\cosh mx-\sinh mx=e^{-mx}$. ....(2)

Hence the above.

Also $d/dx \cosh mx = m \sinh mx$, $d/dx \sinh mx = m \cosh mx$, and equation (1) × (2) gives $\cosh^2 mx - \sinh^2 mx = 1$.

Finally, when $x = 0$ $\cosh mx = 1$, $\sinh mx = 0$.

With these relationships we may proceed.]

Now the end conditions determine the values of the four constants, and since the equation holds *for any value of y*, let us choose a value that we know, namely $y = y_s$, the *static deflexion* of the element.

Consider, say, *a beam supported freely at its ends.*

For this case we know that

$$y_s = 0, \quad \text{when } x = 0 \text{ and } x = L,$$

and since the bending moment $M = EI \dfrac{d^2y_s}{dx^2}$ (see page 281),

$$d^2y_s/dx^2 = 0, \quad \text{when } x = 0 \text{ and } x = L.$$

These four conditions are sufficient to eliminate three of the constants and leave an equation in m, from which the frequencies of vibration are obtained:

$$\frac{p}{2\pi} = \frac{m^2}{2\pi}\sqrt{\frac{EIg}{w}}.$$

We have, differentiating twice,

$$\frac{d^2y_s}{dx^2} = Am^2 \cosh mx + Bm^2 \sinh mx - Cm^2 \cos mx - Dm^2 \sin mx.$$

Thus, using the information above,

$\underline{x = 0}$

$$y_s = 0 = A + C \quad \dots\dots\dots\dots\dots\dots\dots (a)$$

$$\frac{d^2y_s}{dx^2} = 0 = A - C \quad \dots\dots\dots\dots\dots\dots\dots (b)$$

$\underline{x = L}$

$$y_s = 0 = A \cosh mL + B \sinh mL + C \cos mL + D \sin mL \quad \dots (c)$$

$$\frac{d^2y_s}{dx^2} = 0 = A \cosh mL + B \sinh mL - C \cos mL - D \sin mL \quad \dots (d)$$

Hence, from (*a*) + (*b*) and (*a*) − (*b*),

$$A=0 \quad \text{and} \quad C=0\,;$$

$$\therefore \quad (c) \quad B\sinh mL + D\sin mL = 0,$$

$$(d) \quad B\sinh mL - D\sin mL = 0\,;$$

$$\therefore \quad (c)+(d) : B\sinh mL = 0,$$

$$(c)-(d) : D\sin mL = 0.$$

Now since $mL \neq 0$, $\therefore B=0$, and since $D \neq 0$, otherwise $A=B=C=D=0$ and $\therefore y=0$, the beam then being at rest, we have

$$\sin mL = 0\,;$$

$$\therefore \quad \underline{mL = \pi,\ 2\pi,\ 3\pi\ \ldots .}$$

Thus there is an infinite number of values of mL which bear the ratio 1 : 2 : 3 : ... , and as the frequency is given by

$$\frac{m^2}{2\pi}\sqrt{\frac{EIg}{w}},$$

there is an infinite number of modes of vibration, whose frequencies are in the ratio of $1^2 : 2^2 : 3^2 \ldots$.

Taking the lowest value of $m=\frac{\pi}{L}$ we obtain the

$$\text{fundamental frequency} = \frac{\pi}{2}\sqrt{\frac{EIg}{wL^4}}.$$

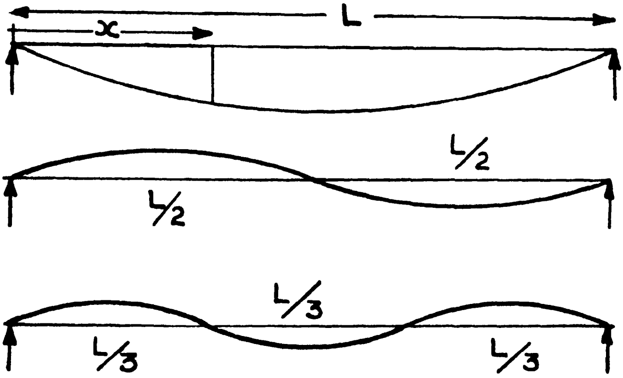


FIG. 102.

Now the equation of the vibration curve is

$$\underline{y = D \sin mx,}$$

and, on substituting the above values of m, π/L, $2\pi/L \dots$, we see that the beam takes up the sine curves shown (Fig. 102) at the different frequencies, consisting of one, two, three, etc., similar bows, thus producing 0, 1, 2, 3 ... nodes between the supports.

As already stated, the values of the constants A, B, C, D in the solution are determined by the end conditions of the beam, so that for the case of a *cantilever carrying a uniformly distributed load*, we have, measuring x from the support :

When $x = 0$, $y_s = 0$ and the slope $\dfrac{dy_s}{dx} = 0$.

When $x = L$, the bending moment $EI\dfrac{d^2y_s}{dx^2}$ is zero, and also the shear force, which is given by $EI\dfrac{d^3y_s}{dx^3}$.

Using this information we can eliminate the constants, and the resulting equation in m will be found to be

$$\cosh mL \,.\, \cos mL = -1.$$

By plotting $\cos mL$ and $-\dfrac{1}{\cosh mL}$ for various values of mL the intersections of the two curves give the various solutions, which are infinite in number, and the corresponding forms of the beam during vibration will be such that 0, 1, 2 ... etc. nodes are produced along its length.

We have examined the one case above in detail in order to emphasise the fact that there is an infinite number of natural modes of transverse vibration of a beam carrying a distributed load, and to call attention to the shape taken up at the different frequencies. Precise mathematical determination of the frequencies becomes extremely difficult for the various complicated systems that occur in practice, and the following approximate method, also due to Lord Rayleigh, enables us to determine, with an error of less than 1 per cent., the *fundamental* frequency for a cantilever or a beam on two supports, however loaded.

The fundamental, being the slowest mode, is usually of most importance to engineers, since it may well come within the operating range of any machine.

Rayleigh's approximate method for determining the fundamental frequency of transverse vibration of a single span beam carrying a distributed load. In this method we consider the energy of the vibrating beam. It will be remembered that the energy is wholly kinetic as the system passes through its mean position, and wholly strain energy at the end of a swing.

Both these terms depend on the displacement of the beam at the point considered, and for their evaluation it will therefore be necessary to know the shape taken up by the beam during vibration. Since the kinetic energy term depends also on the velocity of the S.H.M., by equating the maximum K.E. to the maximum strain energy, the frequency can be determined.

Lord Rayleigh has shown that different assumptions made in regard to the shape, not unreasonably differing from the true curve, hardly affect the calculated frequencies.

A reasonable and very simple assumption is to consider that the displacements of all points in the beam at any instant are the same proportion of their *static displacements* ; or, since the amplitude does not affect the frequency in any way, let us assume that when the beam is at the end of an oscillation the deflexion curve coincides with the static deflexion curve produced by the same loading when the beam is horizontal. The strain energy is then very simply found since the force causing this deflexion is the loading.

That this assumption is reasonable may be seen from the fact that it requires a considerable change in the load-distribution to produce any appreciable change in the static deflexion curve, hence the vibration load-distribution caused by the acceleration of each element would have to be very different from the static to cause much alteration in the shape.

By making such an assumption not strictly correct we are really forcing the beam to take up this given configuration and are therefore introducing an additional constraint, thereby increasing the stiffness of the structure. The values of the frequencies so determined will thus be slightly higher than the true values.

Now let (Fig. 103)

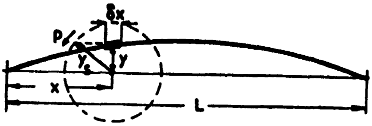


FIG. 103.

w = the weight per unit length of the beam,

L = the length of the beam,

y_s = the static deflexion and amplitude of vibration of an element distant x from one end,

$\frac{p}{2\pi}$ = the frequency of the S.H.M. of each element.

Thus the deflexion at any instant of an element will be

$$y = y_s \cos pt,$$

and

$$\frac{dy}{dt} = -py_s \sin pt.$$

The kinetic energy of any element of width δx is therefore

$$\frac{1}{2}\frac{w\delta x}{g}\left(\frac{dy}{dt}\right)^2,$$

and the maximum value is $\frac{1}{2}\frac{w\delta x}{g}(py_s)^2$, so that the

$$\text{Max. K.E. of whole beam} = \frac{p^2}{2g}\int_0^L wy_s^2 \,.\, dx.$$

The strain energy on our assumption is very easily obtainable, since it will be equal to that stored by the beam when deflected statically under the given loading, and is equal to half the work done by the loads in falling to their static deflexion positions, i.e.

$$\text{Max. strain energy of element} = \frac{1}{2}w\delta x \,.\, y_s;$$

$$\therefore \text{ Max. strain energy of whole beam} = \frac{1}{2}\int_0^L wy_s \, dx.$$

Thus, equating, we obtain :

$$p^2 = \frac{g\int_0^L w y_s \, dx}{\int_0^L w y_s^2 \, dx},$$

or, if the load is *uniformly distributed,*

$$p^2 = \frac{g\int_0^L y_s \, dx}{\int_0^L y_s^2 \, dx},$$

a result which lends itself to ready solution.

In Table 1 the equations for the static deflexion curves for some typical systems are given.

In books on the Strength of Materials it is shown that if

w = the load per unit length,
F = the shear force,
M = the bending moment,
I = the second moment of area of cross-section about the neutral axis,
i = the slope in radians,
y_s = the static deflexion,

then at any section distant x from the origin,

$$\frac{dF}{dx} = w \quad \text{or} \quad F = \int w \, dx + \text{constant};$$

$$\frac{dM}{dx} = F \quad \text{or} \quad M = \int F \, dx + \text{constant};$$

$$\frac{di}{dx} = \frac{M}{EI} \quad \text{or} \quad i = \frac{1}{EI}\int M \, dx + \text{constant};$$

$$\frac{dy_s}{dx} = i \quad \text{or} \quad y_s = \int i \, dx + \text{constant}.$$

[Thus working backwards $\frac{d^2 y_s}{dx^2} = \frac{di}{dx} = \frac{M}{EI}$

$$\therefore \; EI\frac{d^3 y_s}{dx^3} = F \quad \text{and} \quad EI\frac{d^4 y_s}{dx^4} = w.$$]

If the equation of the loading is known, we can at once determine by integration the equation of the static deflexion curve. If there are concentrated loads acting as well, which will cause discontinuities in the mathematical equation for the load, we can only integrate between these loads, or over the portions where the loading is continuous. The constants above are determined by the end conditions of the beam.

As an example let us take the case of a beam carrying a uniformly distributed load and resting on two supports at its ends, which do not constrain it in any manner, e.g. Case (1), Table I.

Then
$$F = w\int dx + C_1$$
$$= wx + C_1.$$

Now when $x = L/2$ the shear force is zero,
$$\therefore\ 0 = \frac{wL}{2} + C_1;$$
$$\therefore\ C_1 = -\frac{wL}{2},$$

and
$$\therefore\ F = wx - \frac{wL}{2};$$
$$\therefore\ M = \int\left(wx - \frac{wL}{2}\right)dx + C_2$$
$$= \frac{wx^2}{2} - \frac{wLx}{2} + C_2.$$

Now when $x = 0$, the bending moment $= 0$; $\therefore\ C_2 = 0$.

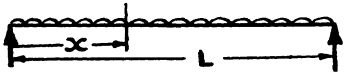
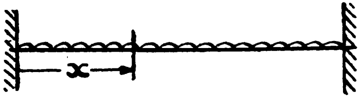
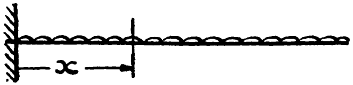
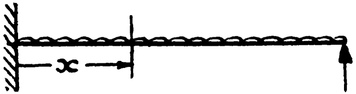
hence
$$i = \frac{1}{EI}\int\left(\frac{wx^2}{2} - \frac{wLx}{2}\right)dx + C_3$$
$$= \frac{w}{2EI}\left(\frac{x^3}{3} - \frac{Lx^2}{2}\right) + C_3.$$

When $x = \frac{L}{2}$ the slope is zero,

so that
$$0 = \frac{w}{2EI}\left(\frac{L^3}{24} - \frac{L^3}{8}\right) + C_3,$$

i.e.
$$C_3 = +\frac{wL^3}{24EI}.$$

TABLE I. BEAMS CARRYING UNIFORMLY DISTRIBUTED LOADS w PER UNIT RUN.

Case	*Ends*	*With origin as shown* y_s	*Approximate value of the fundamental frequency*	*Correct value*	*% Error*
1	 FREE	$\frac{w}{24EI}(x^4-2Lx^3+L^3x)$	$1{\cdot}572\sqrt{\frac{EIg}{wL^4}}$	$\frac{\pi}{2}\sqrt{\frac{EIg}{wL^4}}$	0·06
2	 FIXED	$\frac{w}{24EI}x^2(L-x)^2$	$3{\cdot}57\sqrt{\frac{EIg}{wL^4}}$	$3{\cdot}561\sqrt{\quad}$	0·3
3	 CANTILEVER	$\frac{w}{24EI}(6L^2x^2-4Lx^3+x^4)$	$0{\cdot}5618\sqrt{\quad}$	$0{\cdot}5593\sqrt{\quad}$	0·54
4		$\frac{w}{48EI}(2x^4+3L^2x^2-5Lx^3)$	$2{\cdot}459\sqrt{\quad}$	$2{\cdot}454\sqrt{\quad}$	0·16

Then $$y_s = \frac{w}{2EI}\left(\frac{x^4}{12} - \frac{Lx^3}{6}\right) + \frac{wL^3x}{24EI} + C_4.$$

Finally, when $x = 0$, $y_s = 0$; $\therefore\ C_4 = 0$,

and $$y_s = \frac{w}{24EI}(x^4 - 2Lx^3 + L^3x).$$

Accordingly,

$$p^2 = \frac{g\int_0^L (x^4 - 2Lx^3 + L^3x)\,dx}{\frac{w}{24EI}\int_0^L (x^8 + 4L^2x^6 + L^6x^2 - 4Lx^7 - 4L^4x^4 + 2L^3x^5)\,dx}$$

$$= \frac{g\left[\frac{x^5}{5} - \frac{2Lx^4}{4} + \frac{L^3x^2}{2}\right]_0^L}{\frac{w}{24EI}\left[\frac{x^9}{9} + \frac{4L^2x^7}{7} + \frac{L^6x^3}{3} - \frac{4Lx^8}{8} - \frac{4L^4x^5}{5} + \frac{2L^3x^6}{6}\right]_0^L}$$

$$= \frac{gL^5\left[\frac{1}{5} - \frac{1}{2} + \frac{1}{2}\right]}{\frac{w}{24EI}L^9\left[\frac{1}{9} + \frac{4}{7} + \frac{1}{3} - \frac{1}{2} - \frac{4}{5} + \frac{1}{3}\right]};$$

$$\therefore\ p^2 = \frac{3024}{31}\frac{EIg}{wL^4},$$

or the frequency $= \dfrac{p}{2\pi} = 1{\cdot}572\sqrt{\dfrac{EIg}{wL^4}}$.

Now at the beginning of this chapter we proved that the fundamental frequency for this particular system was

$$\frac{m^2}{2\pi}\sqrt{\frac{EIg}{w}}, \quad \text{where } m \text{ will be } \frac{\pi}{L},$$

i.e. $$\text{true value of frequency} = \frac{\pi}{2}\sqrt{\frac{EIg}{wL^4}}$$

$$= 1{\cdot}571\sqrt{\frac{EIg}{wL^4}},$$

so that the error of the approximate solution is only 0·06 per cent.

The true and approximate values for this fundamental frequency for the systems shown are compared in the last three columns of Table I.

If the beam is of negligible mass, but carries a series of *concentrated loads*, a similar expression will be obtained. Instead of summing up the energies of each element of the beam, we merely sum up the energies of all the loads:

Let $W_1, W_2, W_3 \ldots$ = the concentrated loads;

$y_{s1}, y_{s2}, y_{s3} \ldots$ = the static deflexions of the points of the beam at which the loads are applied—all loads, of course, being in position.

Then,

$$\text{total maximum K.E.} = \frac{p^2}{2g}[W_1 y_{s1}^2 + W_2 y_{s2}^2 + \ldots],$$

and the total maximum strain energy

$$= \tfrac{1}{2}[W_1 y_{s1} + W_2 y_{s2} + \ldots].$$

Hence, equating,

$$p^2 = \frac{g\Sigma W y_s}{\Sigma W y_s^2}.$$

This is the usual method adopted in practice for obtaining the fundamental frequency of transverse vibration of a complicated system. The loading is split up into a series of concentrated loads, which include the weight of the beam sections or other distributed loads, and the static deflexion curve for this system is found by a process of graphical integration. Various methods for obtaining this curve will be found in books on the Strength of Materials, and the following is one of the simplest. The method to be adopted is similar to that given above for obtaining the static deflexion curve by successive integration, but now the integration will be performed graphically. In the mathematical process we can only integrate over portions of the beam on which the load is continuous, but by plotting the curves all discontinuities are automatically included.

Referring to the equations on page 281, if we integrate, say, the Bending Moment equation,

$$M = \int F\,dx + \text{constant}$$

between limits, i.e. between any two points (1), (2) on the beam, we have:

Change of B.M. between these two points $=\int_1^2 F\,dx$, and this is equivalent to *the area of the shear force diagram between these two points.*

Similarly the *change of slope* (radians) between these two points is given by the *area of the B.M. diagram* $\times\frac{1}{EI}$ or, if the cross-section and therefore I varies, a curve of M/I should be constructed, the change of slope then being given by

$$\frac{1}{E}\times \textit{area of } M/I \textit{ curve.}$$

In the same manner, for the deflexion curve we have :

Deflexion of point (2) *from the tangent at* (1) = the area of the slope curve between these two points.

Let us make ourselves familiar with the method by considering a beam carrying, say, three concentrated loads as shown in Fig. 104*a*, the beam being of variable section.

The diagram (*b*) shows the deflexion curve which we are endeavouring to obtain, and is given here so that we can examine certain details as we proceed. Note that α and β are the angles of slope (radians) at each end of the beam, so that $(\alpha+\beta)$, as shown, is the change of slope from A to B. We will call δ_{ab} the deflexion of B from the tangent at A.

Diagram (*c*) shows the shear force, assuming the loads to have the values represented to scale. The B.M. diagram is obtained from this by measuring the areas of the different sections and plotting them as ordinates as in (*d*). For instance, M_{ac} = change of B.M. from A to C = // area on S.F. diagram, and similarly M_{cd} = \\ area. The maximum B.M. obviously occurs where the S.F. changes sign.

The next curve (*e*) is obtained by dividing the ordinates of the B.M. diagram by EI, the resulting form of the curve depending on how the beam section varies, and by integrating this curve we obtain the slope curve (*f*).

The ordinates of this curve represent the *change* of slope from the origin A to the point considered, e.g. the ordinate at $B=(\alpha+\beta)$ = whole area of the M/EI curve.

In order to obtain the actual slope from the horizon, we must subtract from each ordinate the slope at A, i.e. α.

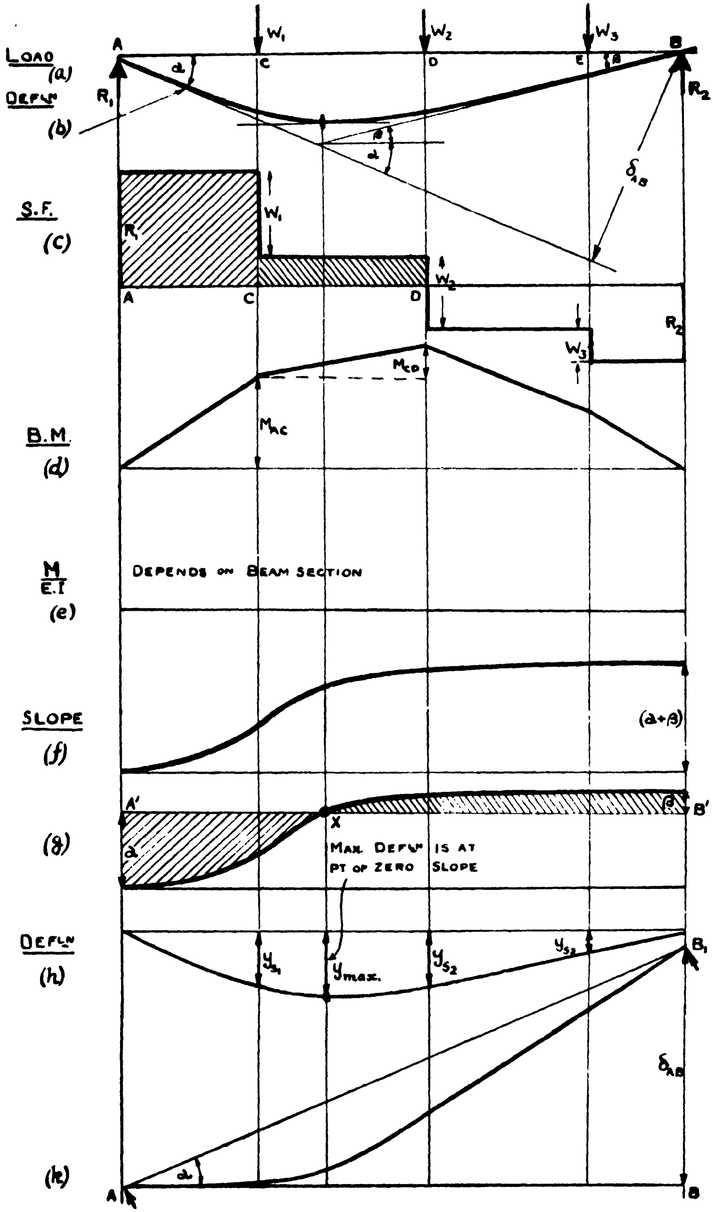

LOAD (a)
DEFL^N (b)
S.F. (c)
B.M. (d)
M/E.I (e)
SLOPE (f)
(g)
DEFL^N (h)
(k)
A
B
C
D
E
W₁
W₂
W₃
R₁
R₂
α
β
δ_AB
M_AC
M_CD
DEPENDS ON BEAM SECTION
(α+β)
A'
B'
X
MAX. DEFL^N IS AT PT OF ZERO SLOPE
y_s₁
y_max.
y_s₂
y_s₃
B₁

FIG. 104.

This has been done in (*g*). At the moment, however, α is unknown, but from this diagram it is easy to see that α is the ***mean height*** of the curve. The slope of the lowest point in the beam, i.e. the point having maximum deflexion y_m, is zero, since the tangent is horizontal, thus the change of slope from this point to $A = \alpha$, and from this point to $B = \beta$. Hence the point X in (*g*) gives this point of maximum deflexion, since, measuring ordinates from this new base line $A'B'$, the ordinate at $A' = \alpha$ and at $B' = \beta$.

Again, the two cross-hatched areas must be equal, since, taking the origin at X, they represent the deflexions of A and B respectively from the tangent at X, i.e. y_m in each case. Thus $\alpha =$ the mean height of the slope curve (*f*). The static deflexion curve (*h*) is now obtained by measuring areas from this new datum line $A'B'$, e.g. area from A' to $X = y_m$.

Instead of transferring our datum and so obtaining actual slopes, we could integrate the curve (*f*) as it is drawn, and would then obtain curve (*k*). The ordinates of this curve represent deflexions *from the tangent at A*, and not from the horizontal, e.g. ordinate at $B = \delta_{ab}$. Now the span of the beam is AB, but if we swing the diagram down and make AB_1 the span, then we have a similar diagram to (*b*), in other words, the static deflexion curve as before. This, of course, depends on α being small so that AB is very nearly equal to AB_1, but the whole deflexion theory is developed on the assumption of small angles and deflexions.

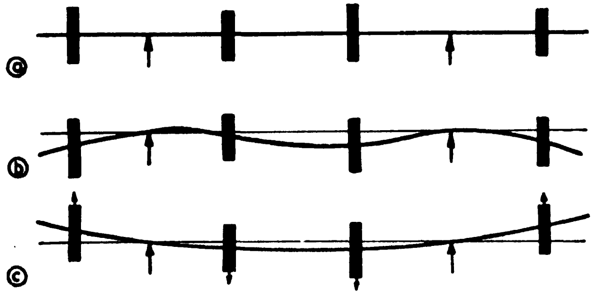


FIG. 105.

There is just one point which might puzzle a student if he wished to determine the fundamental frequency for (say) the system shown in Fig. 105 (*a*). The static deflexion curve would probably be something like that shown in (*b*), but it must be remembered that the reason for obtaining this curve is our assumption that the beam in its *fundamental* vibration takes this shape, and by no stretch of the imagination could it do so in this case. If, however, the loadings on the overhanging portions of the shaft are considered as *negative*, then the shape assumed will be much less constrained and approach more closely the true vibrational form (Fig. 104*c*). Accordingly, all loads on an overhanging portion of a beam should be treated as upward or negative.

Dunkerley's empirical formula for the fundamental frequency of transverse vibration of a loaded beam. This method, devised by Dunkerley in 1893,* consists in splitting up the actual system of loading into a number of simpler systems, the frequencies of which can easily be determined. He showed by experimental verification that the sum of the squares of the reciprocals of these partial frequencies was equal to the square of the reciprocal of the fundamental frequency of the whole system.

For instance, consider the beam, shown in Fig. 106, to carry each load *alone* in turn.

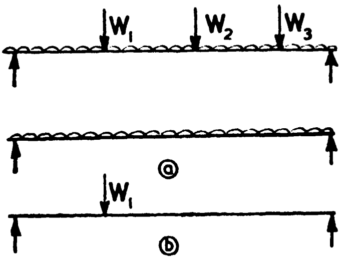


FIG. 106.

Let $\frac{n_0}{2\pi}$ = the frequency of the beam carrying its own weight only (Fig. 106 (*a*)),

$\frac{n_1}{2\pi}$ = the frequency of the beam, mass neglected, carrying the single concentrated load W_1 (Fig. 106 (*b*)),

and similarly,

$\frac{n_2}{2\pi}, \frac{n_3}{2\pi} \dots$ = the frequencies with the loads W_2, W_3 ... when acting alone.

Then, if

$\frac{p_1}{2\pi}$ = the fundamental frequency of the whole system,

$$\frac{1}{p_1^2} = \frac{1}{n_0^2} + \frac{1}{n_1^2} + \frac{1}{n_2^2} + \dots, \qquad \text{(A)}$$

---

* *Phil. Trans. A.*, 1894, vol. clxxxv.

or, if we neglect the mass of the beam,

$$\frac{1}{p_1^2}=\frac{1}{n_1^2}+\frac{1}{n_2^2}+\frac{1}{n_3^2}+\dots, \quad \dots\dots\dots\dots\dots(B)$$

and since

$$\frac{n_1}{2\pi}=\frac{1}{2\pi}\sqrt{\frac{g}{Y_{s1}}},$$

where Y_{s1} is the static deflexion of W_1 *when acting alone*, etc., we have

$$\frac{1}{p_1^2}=\frac{1}{g}[Y_{s1}+Y_{s2}+Y_{s3}+\dots]=\frac{1}{g}\Sigma Y_s.$$

Note the difference between Y_s here and y_s in the previous Rayleigh Formula, which is the static deflexion of a given load *with all loads in action together*.

Also from (A) we see that the stiffer the shaft and therefore the higher its natural frequency $n_0/2\pi$, the less effect it will have on the resultant frequency.

This formula, in general, gives results in fair agreement with the previous method, and can often be employed with advantage.

Professor H. H. Jeffcott, in 1919, in a paper to the Royal Society * has shown the true meaning of this formula, and we will examine this in the next section.

General method for obtaining the frequency equation for all modes of transverse vibration of a beam carrying concentrated loads. The previous formulae enable us to determine with a fair degree of accuracy the *fundamental* frequency of transverse vibration of a loaded beam on not more than two supports, but for the higher frequencies it will be necessary to fall back on the general method for obtaining the frequency equation by writing down the equation of motion for each of the loads.

Consider a single load W_1 in vibration. The equation for its simple harmonic motion is

$$\frac{W_1}{g}\frac{d^2Y_1}{dt^2} \quad \text{or, more concisely,} \quad M_1\ddot{Y}_1=-F_{11}Y_1,$$

where

Y_1 = the displacement of the load at any instant,

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* Derivation of Dunkerley's Empirical Rule, *Proc. Royal Society*, vol. xcv, July 1919.

F_{11} = the stiffness of the beam at the load point, i.e. the force which, when applied at the point of attachment of the load, will produce unit deflexion of the beam at this same point. (The use of the two suffices will be apparent in a moment.)

For our present purpose it will be more convenient to write the equation in the following form

$$Y_1 = -M_1\ddot{Y}_1 \cdot \frac{1}{F_{11}}.$$

Expressing the equation in this manner means that since the force accelerating the load at any instant is $M_1\ddot{Y}_1$, the force exerted on the beam by the load (inertia reaction) is therefore $-M_1\ddot{Y}_1$, which will produce the deflexion Y_1.

If there are other loads at points (2), (3), ... , each of these will cause a deflexion during vibration at the point (1) due to the respective forces $-M_2\ddot{Y}_2$, $-M_3\ddot{Y}_3$, ... which they will exert on the beam.

Let F_{11} = the force which will cause unit deflexion at point (1) when applied at this same point (1)—as already stated,

F_{12}, F_{13} = the forces which will cause unit deflexion at the point (1) when applied respectively at the points (2), (3) ... alone,

then the above inertia reactions of the loads W_2, W_3 ... will produce deflexions at the point (1) of

$$-M_2\ddot{Y}_2 \cdot \frac{1}{F_{12}}, \quad -M_3\ddot{Y}_3 \cdot \frac{1}{F_{13}}, \text{ etc.,}$$

and the total deflexion Y_1 at point (1) will be

$$Y_1 = -M_1\ddot{Y}_1 \cdot \frac{1}{F_{11}} - M_2\ddot{Y}_2 \cdot \frac{1}{F_{12}} - M_3\ddot{Y}_3 \cdot \frac{1}{F_{13}} - \dots.$$

Now since F is the force for unit deflexion, $\frac{1}{F}$ is the deflexion produced by unit force, so that for $\frac{1}{F_{11}}$, $\frac{1}{F_{12}}$, ... we may substitute y_{11}, y_{12} ... , which are the deflexions produced at point (1) by a *unit force* acting alone at points (1), (2) ... respectively. The student may already be familiar with these deflexion coefficients, or "influence numbers" as they are called in "structures" theory. The first suffix indicates the point at which the deflexion is being measured, and the second the point at which the unit force is applied.

Thus,

$$Y_1 = -M_1\ddot{Y}_1 \cdot y_{11} - M_2\ddot{Y}_2 \cdot y_{12} - M_3 Y_3 y_{13} \dots .$$

Similarly, considering point (2) if, y_{21}, y_{22}, y_{23} ... are the deflexions produced at this point by a unit force acting respectively at points 1, 2, 3 ... we have :

$$Y_2 = -M_1\ddot{Y}_1 \cdot y_{21} - M_2\ddot{Y}_2 \cdot y_{22} - M_3\ddot{Y}_3 \cdot y_{23} \dots ,$$

and since the motion is to be simple harmonic of frequency, say, $\frac{p}{2\pi}$ we have, as usual,

$$\ddot{Y}_1 = -p^2 Y_1,$$
$$\ddot{Y}_2 = -p^2 Y_2, \text{ etc.}$$

Thus,

$$Y_1 = M_1 y_{11} p^2 Y_1 + M_2 y_{12} p^2 Y_2 + \dots ,$$
$$Y_2 = M_1 y_{21} p^2 Y_1 + M_2 y_{22} p^2 Y_2 + \dots , \text{ etc.}$$

There are, therefore, x equations for x loads involving x independent variables, i.e. the ratios Y_2/Y_1, Y_3/Y_1 ... etc., and p. Eliminating the Y's, we leave an equation in $(p^2)^x$, the roots of which give the frequencies of the various modes.

Now these equations hold for any type of beam—of uniform or variable section, on any number of supports and with any end conditions—the values of the constants y_{11}, y_{12} ... being determined from these factors by the usual methods to be found in books on "The Strength of Materials".

For the moment we will assume a beam of uniform section arranged on two supports at its ends or as a cantilever, as in our previous discussions.

The values of y_{11}, y_{12} ... in either of these cases can be quickly calculated since, being on a single uniform span, they can be expressed by one formula.

Moreover, Maxwell has shown that if a beam is supported horizontally in any manner, any load attached to a point A will produce the same deflexion at any other point B as would be obtained at A if the same load were placed at B. In other words $y_{ab} = y_{ba}$, etc.

This "reciprocal theorem" is illustrated in Fig. 107 for a beam on three supports and is obviously of great assistance in the calculation of the constants.

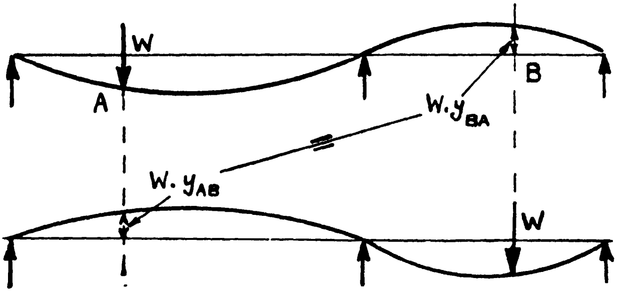


FIG. 107.

Let us, as an example, derive the frequency equation for a beam carrying three concentrated loads.

General Solution. Three Loads (Fig. 108).

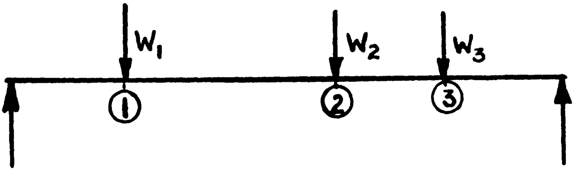


FIG. 108.

The above equations are therefore

$$
\begin{aligned}
&(1)\quad (M_1y_{11}p^2-1)Y_1+M_2y_{12}p^2Y_2+M_3y_{13}p^2Y_3=0,\\
&(2)\quad M_1y_{21}p^2Y_1+(M_2y_{22}p^2-1)Y_2+M_3y_{23}p^2Y_3=0,\\
&(3)\quad M_1y_{31}p^2Y_1+M_2y_{32}p^2Y_2+(M_3y_{33}p^2-1)Y_3=0,
\end{aligned}
$$

or, dividing through by p^2,

$$
\begin{aligned}
&(1)\quad \left(M_1y_{11}-\frac{1}{p^2}\right)Y_1+M_2y_{12}Y_2+M_3y_{13}Y_3=0,\\
&(2)\quad M_1y_{21}Y_1+\left(M_2y_{22}-\frac{1}{p^2}\right)Y_2+M_3y_{23}Y_3=0,\\
&(3)\quad M_1y_{31}Y_1+M_2y_{32}Y_2+\left(M_3y_{33}-\frac{1}{p^2}\right)Y_3=0.
\end{aligned}
$$

The elimination of the Y's is extremely laborious with a large number of loads, but the solution will be considerably accelerated if the reader possesses a knowledge of " determinants ", and in more advanced works such methods are discussed in detail.

Eliminating Y_1, Y_2, Y_3, we have then, expressed as a determinant,

$$\begin{vmatrix} \left(M_1y_{11}-\frac{1}{p^2}\right) & M_2y_{12} & M_3y_{13} \\ M_1y_{21} & \left(M_2y_{22}-\frac{1}{p^2}\right) & M_3y_{23} \\ M_1y_{31} & M_2y_{32} & \left(M_3y_{33}-\frac{1}{p^2}\right) \end{vmatrix}=0$$

which represents in full

$$\begin{aligned}&[p^2]^3M_1M_2M_3\{y_{11}(y_{22}y_{33}-y_{23}^2)+y_{12}(y_{23}y_{31}-y_{12}y_{33})\\ &\qquad\qquad\qquad\qquad+y_{13}(y_{12}y_{32}-y_{22}y_{31})\}\\ &-[p^2]^2[M_1M_2(y_{11}y_{22}-y_{12}^2)+M_2M_3(y_{22}y_{33}-y_{23}^2)\\ &\qquad\qquad\qquad\qquad+M_1M_3(y_{11}y_{33}-y_{13}^2)]\\ &+p^2(M_1y_{11}+M_2y_{22}+M_3y_{33})-1=0.\end{aligned}$$

Now, referring to the algebraical digression on page 85 we see that if p_1^2, p_2^2, p_3^2 represent the roots, then

$$\frac{1}{p_1^2}+\frac{1}{p_2^2}+\frac{1}{p_3^2}=-\frac{a_{n-1}}{a_n}=\frac{M_1y_{11}+M_2y_{22}+M_3y_{33}}{1},$$

and as y_{11} is the deflexion produced at point (1) by a unit load at this point, W_1y_{11} is the *static deflexion* Y_{s1} of the load W_1 *when acting on the beam alone*, and therefore $M_1y_{11}=\frac{Y_{s1}}{g}$.

But the natural frequency of the beam when carrying W_1 only is

$$\frac{n_1}{2\pi}=\frac{1}{2\pi}\sqrt{\frac{g}{Y_{s1}}},$$

hence $$M_1y_{11}=\frac{1}{n_1^2},$$

and similarly $$M_2y_{22}=\frac{1}{n_2^2},$$

where $\frac{n_2}{2\pi}$ is the natural frequency of vibration of the beam when carrying W_2 only, and so on.

Thus, $$\frac{1}{p_1^2}+\frac{1}{p_2^2}+\frac{1}{p_3^2}=\frac{1}{n_1^2}+\frac{1}{n_2^2}+\frac{1}{n_3^2}.$$

A similar result will be found to hold for any number of loads, so that in general

$$\frac{1}{p_1^2}+\frac{1}{p_2^2}+\frac{1}{p_3^2}+\dots=\frac{1}{n_1^2}+\frac{1}{n_2^2}+\frac{1}{n_3^2}+\dots.$$

Now $\frac{p_1}{2\pi}$ gives the fundamental frequency, which is the slowest mode, and the values of p_2, $p_3\dots$, representing the higher harmonics, will, in general, be large in comparison with p_1, so that the values of $\frac{1}{p_2^2}$, $\frac{1}{p_3^2}\dots$ will be negligible, in which case

$$\frac{1}{p_1^2}\doteq\frac{1}{n_1^2}+\frac{1}{n_2^2}+\frac{1}{n_3^2}+\dots.$$

But this is Dunkerley's Formula for the fundamental frequency of the same series of concentrated loads.

Thus its degree of approximation depends on the values of the second and higher frequencies relatively to the first.

[The reader should compare this proof with that for the fundamental mode for a series of loads in torsional oscillation (page 84)].

Let us compare the fundamental frequencies obtained by the three methods described, for the case of a beam of uniform section carrying concentrated loads and freely supported at its ends as shown in Fig. 109*b*.

The deflexion y_x at any point distant x from one support of a uniform beam freely supported at its ends and carrying a single load W (Fig. 109*a*) is given by

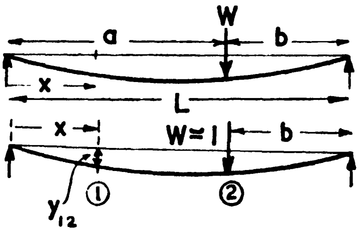


FIG. 109(*a*).

$$y_x=\frac{Wbx}{6EI\,.\,L}(a^2+2ab-x^2),$$

so that if $W=1$ and its position is point (2) whilst point (1) is at the distance x, we have

$$y_{12}=\frac{bx}{6EI\,.\,L}(a^2+2ab-x^2).\quad\dots\dots\dots\dots\dots\dots\text{(i)}$$

This formula can be slightly simplified:

$$y_{12}=\frac{bx}{6EI\,.\,L}\left[(a^2+2ab+b^2)-b^2-x^2\right]$$

$$=\frac{bx}{6EI\,.\,L}(L^2-b^2-x^2).$$

For the evaluation of the influence numbers y_{11}, y_{22}, ... it is perhaps preferable to use equation (i) where $x=a$, the formula reducing to $\frac{a^2b^2}{3EI\,.\,L}$.

All the deflexion coefficients can now be obtained from the above by substituting the respective values of the distances:

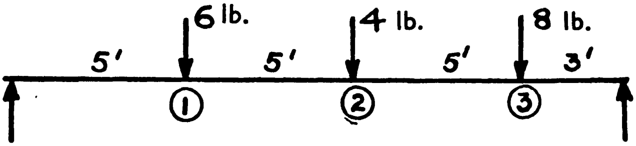
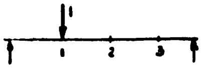


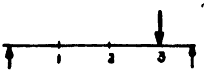
FIG. 109 (*b*).

Thus,



$$\left\{\begin{aligned} y_{11}&=\frac{25\,.\,169}{3EI\,.\,18}=78{\cdot}4\,.\,\frac{1}{EI},\\ y_{12}&=\frac{5\,.\,8(18^2-25-64)}{6EI\,.\,18}=87{\cdot}0\,.\,\frac{1}{EI},\\ y_{13}&=\frac{5\,.\,3(18^2-25-9)}{6EI\,.\,18}=40{\cdot}3\,.\,\frac{1}{EI},\end{aligned}\right.$$

$$\left\{\begin{aligned} y_{21}&=y_{12}=87{\cdot}0\,.\,\frac{1}{EI},\\ y_{22}&=\frac{100\,.\,64}{3\,.\,18\,.\,EI}=118{\cdot}3\,.\,\frac{1}{EI},\\ y_{23}&=\frac{10\,.\,3(18^2-100-9)}{6EI\,.\,18}=59{\cdot}8\,.\,\frac{1}{EI},\end{aligned}\right.$$



$$\left\{\begin{aligned} y_{31}&=y_{13}=40{\cdot}3\,.\,\frac{1}{EI},\\ y_{32}&=y_{23}=59{\cdot}8\,.\,\frac{1}{EI},\\ y_{33}&=\frac{225\,.\,9}{3\,.\,18\,.\,EI}=37{\cdot}5\,.\,\frac{1}{EI}.\end{aligned}\right.$$

Then,

(1) *Rayleigh Method.*

$$p_1^2 = \frac{g[W_1 y_{s1} + W_2 y_{s2} + \ldots]}{[W_1 y_{s1}^2 + W_2 y_{s2}^2 + \ldots]},$$

where $y_{s1} \ldots$ are the static deflexions of the loads *when the beam has all loads on.*

Thus,

$$y_{s1} = W_1 y_{11} + W_2 y_{12} + W_3 y_{13} = \frac{1140\cdot8}{EI},$$

$$y_{s2} = W_1 y_{21} + W_2 y_{22} + W_3 y_{23} = \frac{1473\cdot6}{EI},$$

$$y_{s3} = W_1 y_{31} + W_2 y_{32} + W_3 y_{33} = \frac{781\cdot0}{EI}.$$

Whence, on substituting above we get

$$p_1^2 = \frac{gEI \times 18{,}987}{21\cdot364 \times 10^6};$$

$$\therefore\ p_1 = \underline{0\cdot0298\sqrt{gEI},}$$

or

$$\frac{p_1^2}{gEI} = \frac{1}{1124}.$$

(2) *Dunkerley's Formula.*

$$\frac{1}{p_1^2} = \frac{1}{n_1^2} + \frac{1}{n_2^2} + \frac{1}{n_3^2},$$

where $n_1/2\pi \ldots =$ the frequencies of the beam *carrying the single loads only.*

i.e.

$$\frac{1}{p_1^2} = \frac{1}{g}[W_1 y_{11} + W_2 y_{22} + W_3 y_{33}]$$

$$= \frac{1}{gEI}[(6 \times 78\cdot4) + (4 \times 118\cdot3) + (8 \times 37\cdot5)]$$

$$= \frac{1243\cdot6}{gEI};$$

$$\therefore\ p_1 = \underline{0\cdot0284\sqrt{gEI},}$$

or

$$\frac{p_1^2}{gEI} = \frac{1}{1243\cdot6}.$$

(3) *General Solution. Three Loads.*

The frequency equation has already been deduced on page 294, and on substituting the known values we shall obtain :

$$2{,}140{,}000\left[\frac{p^2}{gEI}\right]^3 - 131{,}770\left[\frac{p^2}{gEI}\right]^2 + 1{,}244\frac{p^2}{gEI} - 1 = 0.$$

The solutions must be obtained by trial and plotting, but in this instance we do know from the previous methods approximately what value to expect for the fundamental so that the work is somewhat simplified, and we shall find that for the fundamental

$$x = \frac{p^2}{gEI} = \frac{1}{1135} = 0{\cdot}00088\,;$$

$$\therefore\ p = 0{\cdot}0296\sqrt{gEI}.$$

Having obtained this one root, the others can be calculated from the quadratic equation resulting by dividing the frequency equation by $(x - 0{\cdot}00088)$, i.e. the original cubic is equivalent to

$$(x - 0{\cdot}00088)[(2140\times 10^3)x^2 - (1299\times 10^2)x + 1138] = 0.$$

Upon solving the quadratic, we get

$$x = 0{\cdot}0106 \quad \text{or} \quad 0{\cdot}05.$$

If we substitute the corresponding values of p^2 in the original equations we shall obtain the values of Y_2 and Y_3 for a given value of Y_1.

For instance, taking the highest frequency,

$$p^2/gEI = x = 0{\cdot}05,$$

and assuming Y_1 = unity, the first two equations (page 293) become

$$17{\cdot}4Y_2 + 16{\cdot}12Y_3 = -22{\cdot}52,$$
$$22{\cdot}66Y_2 + 23{\cdot}92Y_3 = -26{\cdot}1,$$

and therefore

$$Y_2 = -2{\cdot}34,$$
$$Y_3 = +1{\cdot}13.$$

Since Y_2 is of opposite sign to Y_1 and Y_3, the vibration curve consists of three bows producing two nodes between the loads. Similarly, for the fundamental all three signs will be the same showing that the configuration is a single bow, whilst for the intermediate frequency the signs will be such as to produce two bows.

Transverse vibrations of a tightly stretched wire carrying a series of concentrated loads, Fig. 110 (*a*). Whilst not so important to engineers, this problem nevertheless is of interest due to the ease with which the influence numbers can be obtained.

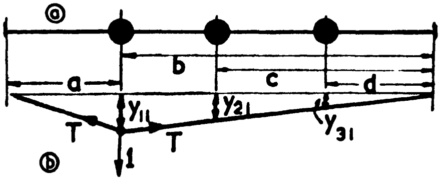


FIG. 110.

The tension T in the wire is assumed so great that it may be considered constant when the wire suffers small deflexions during vibration.

Consider a unit load placed at point (1), producing therefore a deflexion y_{11} at this point (Fig. 110 (*b*)). Since this deflexion is assumed small compared with the span and the two sections of the wire remain straight when deflected, we can say that the vertical component of the tension T in the length a is $\frac{y_{11}}{a}T$, whilst that due to the length b is $\frac{y_{11}}{b}T$.

The sum of these two components must balance the unit force which is being applied, i.e.

$$y_{11}\left(\frac{1}{a}+\frac{1}{b}\right)T=y_{11}\left(\frac{L}{ab}\right)T=1,$$

or

$$y_{11}=\frac{1}{T}\cdot\frac{ab}{L},$$

and it will be seen that

$$y_{12}=y_{21}=y_{11}\cdot\frac{c}{b},$$

$$y_{13}=y_{31}=y_{11}\frac{d}{b},\ \text{etc.},$$

so that the equation of motion for the first mass M_1 will be

$$Y_1=p^2y_{11}\left[M_1Y_1+M_2\cdot\frac{c}{b}\cdot Y_2+M_3\cdot\frac{d}{b}\cdot Y_3+\ldots\right],$$

or

$$0=\left(M_1-\frac{1}{p^2y_{11}}\right)Y_1+M_2\cdot\frac{c}{b}\cdot Y_2+\ldots,$$

and the equations for the remainder of the masses can be built up in a similar manner.

Transverse vibration of beams on more than two supports. So far we have considered methods for determining the frequencies of tranverse vibration of cantilevers and beams on two supports, but this book would be incomplete without some discussion on continuous beams. Various methods have been devised and can easily be applied to simple cases, but the solution becomes very involved when the number of loads and supports is large.

Consider a simple system such as shown in Fig. 111—two concentrated loads on a beam of negligible mass, and let us for our purpose imagine it to be a shaft supported in three bearings.

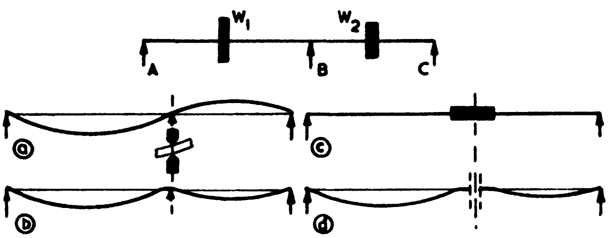


FIG. 111.

Two natural modes of transverse vibration can occur, but everything depends on the type of support or bearing B.

If the bearing is *short*, and may therefore be assumed to produce no bending moment on the shaft, not constraining it in any way other than producing a node at that point, then the shaft could take up either of the shapes shown (*a*), (*b*) during vibration.

The fundamental mode is given by (*a*), since it is the most natural shape for the beam to assume, there being a smooth sweep of the deflexion curve from one section to the other, each section, as it were, assisting the deflexion of the other.

The second mode (*b*) is obviously constrained and would represent a higher frequency.

If, however, the bearing B is *long*, i.e. completely constrains the shaft to be horizontal at this section, then we could imagine the shaft cut at this bearing without affecting matters, since neither section, as regards its deflexion, can affect the other. Hence the frequencies would be those of each load on its own section of shafting as indicated in Fig. 111*c*, *d*.

In many steam turbine designs long bearings are introduced to afford some constraint, but even with such a bearing there must be a certain amount of clearance, and it would be difficult to say exactly what degree of constraint was introduced. The amount will depend on the deflexion, and as only a slight deflexion will cause fouling of the blading with the casing, considerable damage might be done before any constraint could be exercised by the bearing. Hence in the calculation of the natural frequencies of transverse vibration of such a system it would probably be safer to assume short bearings throughout, thus giving the lowest values.

We shall assume short bearings at all intermediate bearings, and will consider two methods of attack, the first by direct calculation, the second by a semi-graphical process.

(1) In discussing the general method for obtaining the frequency equation for a loaded beam, we saw that it was applicable to *any* beam arrangement, the influence numbers being calculated to suit. For instance, for two loads on two spans as indicated in Fig. 111 we have as before

$$Y_1 = +M_1 y_{11} p^2 Y_1 + M_2 y_{12} p^2 Y_2,$$

$$Y_2 = +M_1 y_{21} p^2 Y_1 + M_2 y_{22} p^2 Y_2,$$

the values of y_{11}, y_{12} ... being calculated by the ordinary methods for continuous beams. Alternatively, if the intermediate bearing is short, i.e. exerts no bending moment on the beam but only prevents deflexion at that point, then the reaction of the support can be considered as a negative load, acting, therefore, in the same direction as the inertia reactions of the actual loads, and the beam may be imagined as supported *at its ends only* and under the action of these forces. This, of course, holds for any number of intermediate supports, ***and the influence numbers will now be obtained from the simple formulae applicable to a single span beam.*** For instance, at any point x along a beam

(1) if there is a load W_x, this will cause a deflexion at any other point z

$$= -M_x \ddot{Y}_x y_{zx},$$

as we have seen, and

(2) if there is a support exerting an upward reaction R_x, this will cause a negative deflexion at z

$$= -R_x y_{zx},$$

and at the same time, the total deflexion at x, i.e. Y_x will equal zero, assuming the supports are at the same level.

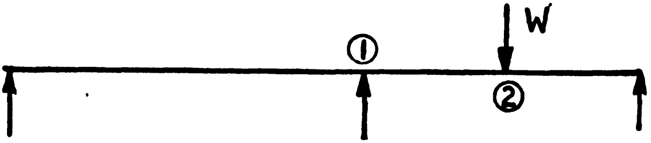


FIG. 112.

Consider, say, a beam on three supports and carrying a single load (Fig. 112). We can imagine this as a beam supported at its ends and carrying two loads, $-R_1$ at the point (1) and $-M_2\ddot{Y}_2$ at (2), while the resultant deflexion Y_1 at (1) is zero. Hence

$$0 = Y_1 = -R_1 y_{11} - M_2 y_{12} \ddot{Y}_2,$$

$$Y_2 = -R_1 y_{21} - M_2 y_{22} \ddot{Y}_2.$$

Assuming, as before, S.H.M., $\ddot{Y}_2 = -p^2 Y_2$, we have

$$0 = -R_1 y_{11} + M_2 y_{12} p^2 Y_2 \quad \dots\dots\dots\dots\dots (a)$$

$$Y_2 = -R_1 y_{21} + M_2 y_{22} p^2 Y_2$$

or

$$0 = -R_1 y_{21} + (M_2 y_{22} p^2 - 1) Y_2 \quad \dots\dots\dots\dots\dots (b)$$

There is no need to determine the value of R, since it will be seen that R_1 replaces Y_1, and there will always be enough equations to eliminate both the R's and Y's.

Eliminating R_1 and Y_2 from (a) and (b)

$$\begin{vmatrix} -y_{11} & M_2 y_{12} p^2 \\ -y_{21} & (M_2 y_{22} p^2 - 1) \end{vmatrix} = 0,$$

i.e.

$$M_2 p^2 (y_{11} y_{22} - y_{12}^2) - y_{11} = 0.$$

Similarly, for any number of short bearings, we may write down the equations and by eliminating the R's and Y's obtain the frequency equation for all the possible modes.

Let us take as a further example the system illustrated in Fig. 113. This is the same arrangement as Fig. 109, but load (2) has been replaced by a support.

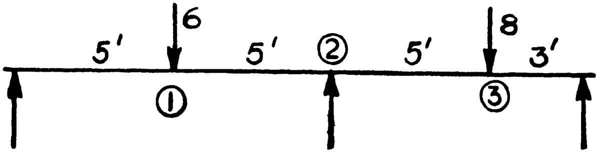


FIG. 113.

In general terms then we have

$$Y_1 = M_1 y_{11} p^2 Y_1 - R_2 y_{12} + M_3 y_{13} p^2 Y_3,$$
$$0 = M_1 y_{21} p^2 Y_1 - R_2 y_{22} + M_3 y_{23} p^2 Y_3,$$
$$Y_3 = M_1 y_{31} p^2 Y_1 - R_2 y_{32} + M_3 y_{33} p^2 Y_3.$$

Hence

$$0 = (M_1 y_{11} p^2 - 1) Y_1 - R_2 y_{12} + M_3 y_{13} p^2 Y_3,$$
$$0 = M_1 y_{21} p^2 Y_1 - R_2 y_{22} + M_3 y_{23} p^2 Y_3,$$
$$0 = M_1 y_{31} p^2 Y_1 - R_2 y_{32} + (M_3 y_{33} p^2 - 1) Y_3.$$

Eliminating Y_1, R and Y_3, we have

$$\begin{vmatrix} (M_1 y_{11} p^2 - 1) & -y_{12} & M_3 y_{13} p^2 \\ M_1 y_{21} p^2 & -y_{22} & M_3 y_{23} p^2 \\ M_1 y_{31} p^2 & -y_{32} & (M_3 y_{33} p^2 - 1) \end{vmatrix} = 0$$

i.e. in full,

$$[p^2]^2 M_1 M_3 [y_{11} y_{32} y_{23} - y_{11} y_{22} y_{33} + y_{12} y_{21} y_{33} - y_{12} y_{23} y_{31}$$
$$+ y_{13} y_{22} y_{31} - y_{13} y_{21} y_{32}]$$
$$+ p^2 [M_1 (y_{11} y_{22} - y_{12}^2) + M_3 (y_{22} y_{33} - y_{23}^2)]$$
$$- y_{22} = 0.$$

Whence, on substituting the known values, we shall obtain

$$535{,}000 \left[\frac{p^2}{gEI}\right]^2 - 17{,}200 \left[\frac{p^2}{gEI}\right] + 118 \cdot 3 = 0;$$

$$\therefore \frac{p^2}{gEI} = 0 \cdot 0221 \quad \text{or} \quad 0 \cdot 00996.$$

Method (2) *Semi-Graphical.*

In this method we assume some initial shape for the deflexion curve taken up by the beam during vibration which will satisfy the end conditions, and also a value for the frequency $\frac{p}{2\pi}$.

Let Y_1, $Y_2 \ldots$ be the deflexions of the loads W_1, $W_2 \ldots$ so assumed, so that the inertia reactions will be

$$M_1p^2Y_1,\ M_2p^2Y_2,\ \ldots \text{ as usual.}$$

These reactions are now considered as forces statically applied to the beam, and the static deflexion curve for this new system is obtained graphically or otherwise.

If our assumption about the initial shape is correct, the deflexions Y_1', $Y_2' \ldots$ which result will be proportional to the deflexions Y_1, Y_2 initially assumed, and the true frequency will be given by $\frac{p'}{2\pi}$, where

$$p'^2Y' = p^2Y \quad \text{or} \quad p' = p\sqrt{\frac{Y_1}{Y_1'}}, \quad = p\sqrt{\frac{Y_2}{Y_2'}} \text{ etc.,}$$

since, if we use this new value p' for the frequency in calculating our inertia loads, all the forces, and therefore the deflexions, will be increased in the ratio $\frac{Y}{Y'}$ i.e. the deflexion curve, obtained graphically will coincide with the one initially assumed.

Suppose, however, that on working out any given example for the *fundamental mode,* it is not considered that the ratios Y_1/Y_1', Y_2/Y_2', etc., are near enough the same, i.e. we have not been fortunate in our initial assumption of the shape of the curve, then Professor A. Stodola * has shown that if we start again using now the deflexions Y_1', $Y_2' \ldots$ as the assumed shape for calculating the inertia loads, we shall arrive at a result more closely approximating to the correct, and further successive approximations converge to the correct value. In general it will be found that at the most one further approximation only will be required.

Let us take, as a simple example, the system considered in the previous method.

---

* *Steam Turbines* (Constable & Co.)

The actual loading diagram is reproduced in Fig. 114 (*a*). We will determine the value of the fundamental frequency, the beam during vibration then assuming some such shape as in (*b*) as we have previously pointed out. For this case then the inertia loads due to W_1 and W_3 must be considered as acting in opposite directions, and here that due to W_3 has been termed negative.

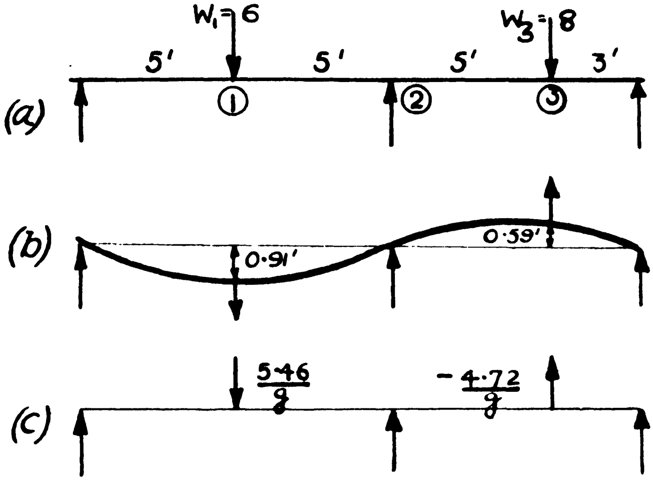


FIG. 114.

The deflexion curve assumed in (*b*) has been made up from the arcs of two circles, and from this by measurement in the original:

$$Y_1 = 0{\cdot}91 \text{ ft.},$$

$$Y_3 = 0{\cdot}59 \text{ ft.}$$

Let us assume p = unity, then the inertia loads are

$$(1)\ \frac{6}{g} \times 0{\cdot}91 \qquad (3)\ -\frac{8}{g} \times 0{\cdot}59,$$

i.e.

$$\frac{5{\cdot}46}{g} \qquad -\frac{4{\cdot}72}{g}.$$

Thus Fig. 114 (*c*) gives the new loading system for which the static deflexions Y_1', Y_3' are required.

This problem may now be solved by any of the various methods for determining the deflexion of loaded beams that may be known to the reader.

A simple method in this case is to consider the beam supported at its ends and under the action of the forces (1), (3) and the reaction R_2 separately. The deflexions Y_1, Y_2, Y_3 are then determined being as before the sum of the deflexions due to each single load, while the necessary value of R_2 is obtained from the fact that $Y_2=0$.

In other words, being given the value of p, we are solving for Y_1, Y_3, etc., in the equations on page 303.

As before,

$$\text{(1)}\ \left.\begin{aligned} y_{11}&=78{\cdot}4/EI\\ y_{12}&=87/EI\\ y_{13}&=40{\cdot}3/EI\end{aligned}\right\}\quad \begin{array}{c}\therefore\ \text{for actual}\\ \text{deflexions due}\\ \text{to load (1)}\\ \text{multiply by } 5{\cdot}46/g\end{array} = \left\{\begin{array}{ll}(1) & 428/EIg\\ (2) & 475/EIg\\ (3) & 220/EIg\end{array}\right.$$

$$\text{(2)}\ \left.\begin{aligned} y_{21}&=87/EI\\ y_{22}&=118{\cdot}3/EI\\ y_{23}&=59{\cdot}8/EI\end{aligned}\right\}\quad \begin{array}{c}\text{Multiply by}\\ -R_2\\ \text{when found}\end{array} = \left\{\begin{array}{ll}(1) & -\mathit{142/EIg}\\ (2) & -\mathit{193/EIg}\\ (3) & -\mathit{97.5/EIg}\end{array}\right.$$

$$\text{(3)}\ \left.\begin{aligned} y_{31}&=40{\cdot}3/EI\\ y_{32}&=59{\cdot}8/EI\\ y_{33}&=37{\cdot}5/EI\end{aligned}\right\}\quad \begin{array}{c}\text{Multiply by}\\ -4{\cdot}72/g\end{array} = \left\{\begin{array}{ll}(1) & -190/EIg\\ (2) & -282/EIg\\ (3) & -177/EIg\end{array}\right.$$

Then to find R_2 we know that $Y_2=0$, i.e.

$$\frac{475}{EIg}-\frac{118{\cdot}3}{EI}\cdot R_2-\frac{282}{EIg}=0\,;$$

$$\therefore\ R_2=\frac{1{\cdot}63}{g}\,.$$

The values for the deflexions due to R_2 can now be obtained and are given in italics above.

$$\therefore\ Y_1'=\frac{1}{EIg}[428-142-190]=\frac{96}{EIg},$$

and $$Y_3'=\frac{1}{EIg}[220-97{\cdot}5-177]=-\frac{54{\cdot}5}{EIg}\,.$$

Hence, if our original curve is correct,

$$\text{True value} \qquad p'^2 = p^2 \frac{Y_1}{Y_1'} = p^2 \frac{Y_3}{Y_3'};$$

$$\therefore \text{ For point (1)} \qquad p'^2 = 1 \times \frac{0{\cdot}91}{96} EIg = \underline{0{\cdot}00949\, EIg,}$$

$$\text{and point (2)} \qquad p'^2 = 1 \times \frac{0{\cdot}59}{54{\cdot}5} EIg = \underline{0{\cdot}01083\, EIg\,;}$$

$$\therefore \text{ Mean value} \qquad p'^2 = \underline{0{\cdot}0101\, EIg.}$$

By previous calculation we obtained

$$p_1^2 = \underline{0{\cdot}00996\, EIg.}$$

Suppose that these values are not considered near enough and that a second approximation should be performed as described above.

We assume, then, that the deflexions Y_1', Y_3' just obtained are our initial values for determining the inertia loads

$$Y_1' = 96/EIg,$$

$$Y_3' = -54{\cdot}5/EIg.$$

Also a value for p^2 has to be assumed.

It will be convenient to take

$$p^2 = \frac{EIg}{100},$$

so that our inertia loads become

$$\frac{6}{g} \times \frac{EIg}{100} \times \frac{96}{EIg} = \frac{5{\cdot}76}{g},$$

and
$$-\frac{8}{g} \times 0{\cdot}545 = -\frac{4{\cdot}36}{g}.$$

Using these loads and proceeding as before, we obtain

$$Y_1'' = \frac{95{\cdot}7}{EIg},$$

$$Y_3'' = -\frac{55{\cdot}5}{EIg};$$

$$\therefore\ p''^2 = p'^2 \cdot \frac{Y_1'}{Y_1''}$$

$$= \frac{EIg}{100} \cdot \frac{96}{95{\cdot}7} = \underline{0{\cdot}01002\, EIg,}$$

or, from load (2),

$$p''^2 = \frac{EIg}{100} \cdot \frac{54{\cdot}5}{55{\cdot}5} = \underline{0{\cdot}00984\, EIg\,;}$$

$$\therefore\ \underline{\text{Mean } p''^2 = 0{\cdot}00993\, EIg.}$$

With beams of varying cross-section and having a large number of loads and supports, a graphical method for obtaining the static deflexion curve, as previously described, will be of great aid. The reactions at the supports would first be obtained, and then the shear force or bending moment diagram can be constructed.

CHAPTER XIII

THE WHIRLING OF SHAFTS

In this chapter we propose to discuss various theories put forward to explain the behaviour of a shaft at high speeds of rotation.

If the revolutions of an unloaded shaft are gradually increased, it will be found that a certain speed will be reached at which violent instability will occur, the shaft deflecting into a single bow and whirling round like a skipping-rope. If this speed is maintained the deflexion will become so large that the shaft will be fractured, but if this speed is quickly run through the shaft will become straight again and run true until at another higher speed the same phenomena will occur, the deflexion now, however, being in a double bow—and so on. Such speeds are called *critical speeds of whirling.*

Professor Rankine was the first to attempt an explanation of the behaviour of an unloaded shaft, and Professor Osborne Reynolds carried investigations a stage further by considering a shaft carrying concentrated loads ; but it was not until 1884 that general interest in the phenomena was aroused by the publication of two papers by Dunkerley, the first being communicated to the *Phil. Trans.* and the second, a few months later, to the Liverpool Engineering Society.

Owing to non-homogeneity of the material, practical difficulties, etc., it is impossible to make the centre of gravity of any section of shaft or of the attached masses coincide exactly with the axis of rotation, hence centrifugal forces will be set up tending to deflect the shaft away from its rest position. These centrifugal forces are balanced only by the elastic righting forces of the shaft, and Dunkerley equated the two and deduced from the equation the speeds at which the shafting would suffer an infinite deflexion.

For our purpose we will consider the simplest case of all, namely a single heavy wheel or disk on a light spindle, the inertia of which can therefore be neglected.

A well-known example of this is the rotor of the de Laval turbine, the introduction of which was the real cause of a general interest being taken in this subject. As before, we will neglect gravity, considering the axis through the bearings as the rest position of the shaft. Thus in Fig. 115:

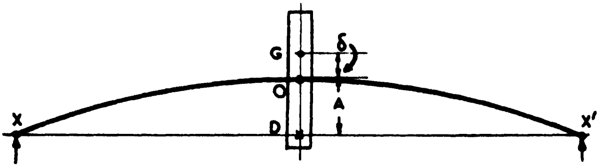


Fig. 115.

Let δ = the eccentricity of the C.G. from the wheel centre O,

A = the deflexion of O from the rest position D due to the centrifugal force set up at any given speed ω.

Then the centre of gravity G revolves round the point D in a circle of radius $(A+\delta)$, and the centrifugal reaction exerted is

$$\frac{W}{g}\omega^2(A+\delta).$$

This must be balanced by the elastic righting force of the shaft $= FA$, where, as usual,

F = the force to produce unit deflexion at the point O, i.e. the stiffness of the shaft at this point.

Hence

$$\frac{W}{g}\omega^2(A+\delta)=FA;$$

$$\therefore\ A=\frac{\frac{W}{g}\omega^2 . \delta}{F-\frac{W}{g}\omega^2}. \qquad \text{.......................(1)}$$

From this equation we see that A will become infinite when

$$\left(F-\frac{W}{g}\omega^2\right)=0, \quad \text{or} \quad \omega^2=\frac{Fg}{W}.$$

Thus, if ω_c denotes the critical whirling speed at which fracture will occur due to excessive deflexion,

$$\omega_c = \sqrt{\frac{Fg}{W}} \text{ radians per sec.,}$$

or if N_c = the whirling speed in *revs. per sec.*,

$$N_c = \frac{1}{2\pi}\sqrt{\frac{Fg}{W}}.$$

But this is the same expression as was obtained for the *natural frequency of transverse vibration* of the system.

For other cases also, Dunkerley deduced that the whirling speeds (revs. per sec.) were equal to the natural frequencies of transverse vibration, there being the same number of whirling speeds as natural frequencies for a given system, the shaft taking up a similar shape for each whirling speed as it would if in transverse vibration.

Let us return to equation (1). This may be written

$$A = \frac{\omega^2}{\omega_c^2 - \omega^2} \cdot \delta. \quad \text{........................(2)}$$

We deduce from (2) that :

When the running speed ω is less than the critical value ω_c, *A and δ have the same sign, i.e. the centre of gravity G is situated from D on the further side of O.* Fig. 115 shows this condition for slow speed.

As the speed ω increases, the deflexion A increases, becoming infinite when $\omega = \omega_c$, as we have noted, but if we increase the speed above this value A and δ *are now of opposite sign, which means that the centre of gravity G is now situated between D and O,* as shown in Fig. 116. As ω continues to increase A now diminishes, until finally at a very high speed when ω_c is negligible compared with ω, A approaches the value $-\delta$, i.e. G coincides

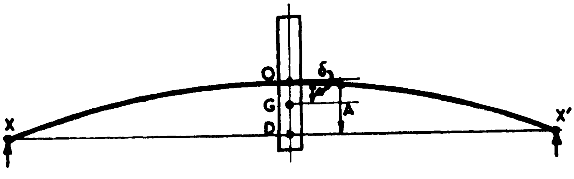


FIG. 116.

with D and the wheel rotates about its centre of gravity with perfect stability. Upon this principle the successful running of the de Laval rotor depends, the operating speed being about seven times the whirling speed, and in running up the machine to speed through the whirling speed, the critical conditions will exist for so short a time that dangerous deflexions will not occur.

The reader will probably have detected a complete similarity between equation (2) and our deductions therefrom and the solution of the equation of motion for a single load W undergoing a simple harmonic *forced vibration*.

Thus after the publication of Dunkerley's results, other investigators—notably Dr. Chree *—endeavoured to explain the phenomenon of whirling as due to forced vibrations, and the sequence of papers published on the subject make most interesting reading.

That these phenomena do occur can easily be demonstrated by illuminating stroboscopically a model rotor which is set out of balance by attaching a small mass to the disk, so that for all practical purposes the centre of gravity will lie on a radial line passing through this mass.

A good many years ago the author, at the suggestion of Professor W. J. Kearton, built such a model, and, whilst it demonstrated wonderfully well the truth of the deductions made above, another totally unexpected result was made apparent, namely that *at the whirling speed a radial line passing through the attached mass was at right angles to the plane of bending of the shaft, i.e.* A *and* δ *were at right angles.*

As the shaft was run up in speed the attached mass appeared to travel from the plane of bending, as shown in Fig. 115, to this position, and then, on passing the critical speed, to travel on round through another 90 degrees into the plane of bending once more, but within the bow as in Fig. 116.

In the above theory it has obviously been tacitly assumed that A and δ are *in the same straight line*, the centrifugal force which has been causing the deflexion being in the same plane as the bent shaft, but at the whirling speed when this deflexion is greatest this apparently is not so.

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* *Phil. Mag.*, 1904, vol. 7.

Now we have already noted a similarity between our deductions and those previously obtained for a forced vibration, and here again another similarity occurs, since it will be remembered that when resonant conditions exist for a forced vibration, the disturbing force vector is 90 degrees in advance of the displacement vector.

It was left to Professor H. H. Jeffcott to supply the clue to this similarity in his paper " The Lateral Vibration of Loaded Shafts in the Neighbourhood of a Whirling Speed " (*Phil. Mag.*, 1919), and a few years later Frith and Buckingham discussed the whirling of a de Laval rotor in their paper " The Whirling of Shafts " (*Journal of Electrical Engineers*, 1924), and this new theory gives a satisfactory explanation of the observed phenomena.

We have always assumed that the shaft when deflected revolves bodily about the axis of its bearings. Jeffcott, however, maintained that each element of a shaft revolves not bodily about this axis *but about its own axis.* Adopting this view, the phenomenon can be explained as due to forced vibrations.

Attention has already been called to the fact that if a body, say an element of a shaft, is given two simple harmonic motions at right angles,

$$x_1 = A \cos \omega t,$$

$$x_2 = A \sin \omega t,$$

of the same amplitude and phase, then it will perform a circular motion of radius A with angular velocity ω (Fig. 117).

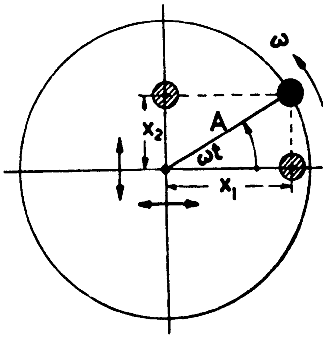

FIG. 117.

Thus, if we impose two such transverse vibrations on a shaft, it will perform this circular vibration with angular velocity ω, but it will be clear from an examination of Fig. 118 (a) that the shaft does not rotate in a fixed bent position (although to the eye it would appear to do so) since the vertical and horizontal axes through the points A and B respectively will *remain parallel to the original axes, there being a fluctuation in strain round the section.*

But if we cause each element of the shaft to rotate about its own centre as well, at the same speed ω as that of the circular vibration, the shaft no longer suffers a fluctuation in strain but takes up a definite bent position (Fig. 118*b*).

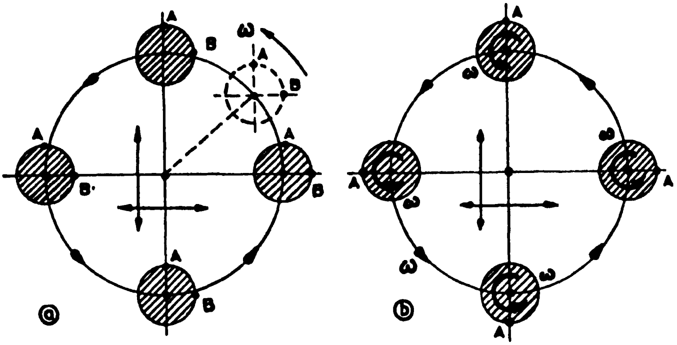

FIG. 118.

Shows in end view the circular vibration of any element of the shaft, the positions depicted being those at the ends of the horizontal and vertical component tranverse vibrations.

Consider again a de Laval rotor, whirling may be explained in this manner : *as a rotation of the disk about its own centre with angular velocity* ω, *producing a centrifugal force* $\frac{W}{g}\omega^2\delta$, the components of which resolved on two planes at right angles will cause two simple harmonic forced vibrations as above, thus causing a rotation about the rest position at radius A with angular velocity ω, e.g. :

Taking vectors for a forced vibration as before, since $\delta =$ the eccentricity of the centre of gravity G, this will represent in direction the maximum value of the disturbing force

$$P=\frac{W}{g}\omega^2\delta=oc$$

for any given value of ω (Fig. 119). This force may be split up into two S.H. forces at right angles :

$\frac{W}{g}\omega^2\delta \,.\, \cos \omega t$ forcing a vibration in the vertical plane,

$\frac{W}{g}\omega^2\delta \,.\, \sin \omega t$ forcing a vibration in the horizontal plane.

Consider (say) the vibration produced in the vertical plane :

If A is the amplitude we have, as before,

(1) *No Damping* :

$$P = \text{vector } oc = oa - ob = FA - \frac{W}{g}\omega^2 A,$$

and is in phase with the displacement.

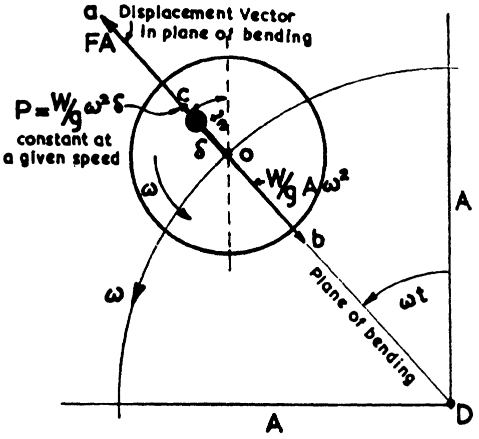
a
Displacement Vector in plane of bending
FA
P=W/g ω²δ
constant at a given speed
c
δ
o
W/g Aω²
ω
b
A
Plane of bending
ωt
ω
A
D

FIG. 119.

Similarly for the vibration in the horizontal plane, the combination of the two causing a circular vibration of radius A about the rest position D, which together with the rotation of the disk about its own centre produces rotation of the shaft in a fixed bent position.

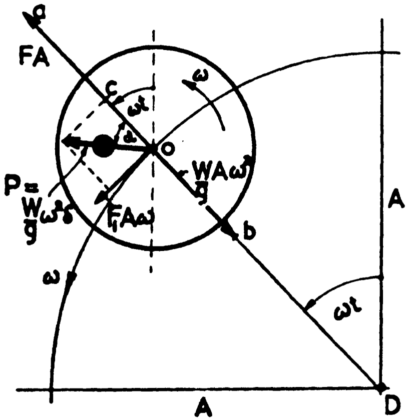

FIG. 120.

Thus, for either vibration,

$$P = \frac{W}{g}\omega^2\delta = FA - \frac{W}{g}\omega^2 A,$$

$$\text{or } \frac{W}{g}\omega^2(A+\delta) = FA. \quad \ldots(1)$$

It will be seen that A and δ are here in the same straight line and equation (1) is that obtained by Dunkerley.

(2) *With Damping* (Fig. 120).

The disturbing force vector is now at an angle α to the displacement vector, ***i.e. A and δ are no longer in same straight line,*** and

$$\frac{W}{g}\omega^2\delta \,.\, \cos\alpha = FA - \frac{W}{g}\omega^2 A.$$

When resonance occurs, i.e.

$$\omega = \frac{Fg}{W},$$

$$\alpha = 90^\circ,$$

i.e. *A and δ are at right angles.*

The amplitude A is very large, being infinite for no damping, and this is the speed termed the whirling speed, being that corresponding to coincidence of the forced and natural frequencies. Fig. 121 shows these conditions, which explains the observed phenomenon mentioned above.

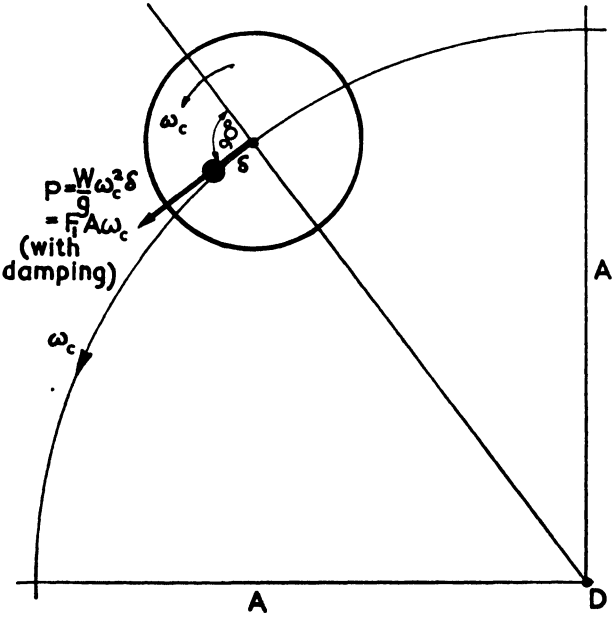

Fig. 121.

Fig. 122 represents the conditions above the critical speed. The amplitude and radius of rotation A diminishes as ω increases, and α approaches 180°, when finally $A = -\delta$.

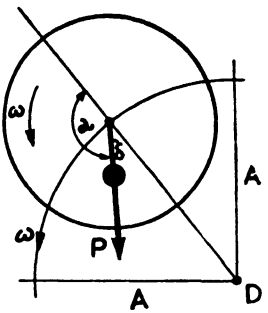

FIG. 122.

Thus assuming no damping, we see that A and δ are in the same straight line and the equation of the previous method happens to give the right result. There would be a sudden change of A and δ to the 90° position when the critical speed was reached. In the experiment mentioned above it was stated that the weight appeared to move slowly round relative to the plane of bending, in other words there was considerable damping, the angle α therefore gradually increasing. The damping occurred owing to the necessity of holding the shaft during rotation to prevent excessive amplitude.*

Recapitulating: we may define the whirling of a de Laval rotor as the resonant conditions of a forced circular vibration caused by the two simple harmonic components at right angles of the centrifugal force set up by the disk revolving about its own centre.

When we consider a series of disks or a distributed load, it is admittedly more difficult to visualise what actually occurs; each shaft element or disk must be assumed to be rotating about its own centre producing a centrifugal force, and the resultant of all such forces may be taken as forcing the circular vibration of the whole shaft.

Thus to find the whirling speed of any system, we merely have to determine the natural frequency of transverse vibration of the same system.

The number of whirling speeds of a given system is obviously equal to the number of modes of transverse vibration, the shaft taking up similar shapes.

In the above examination we have seen that the shaft whirls in a fixed bent position, since the rotation of the shaft about its own axis is in step with the circular vibration about its rest position.

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* See Appendix B.

It is interesting to note that B. L. Newkirk has shown that it is possible to force a circular vibration of angular velocity equal to the whirling ω_c when the shaft is rotating about its own axis at about twice this speed, if the shaft runs in perfectly lubricated bearings.*

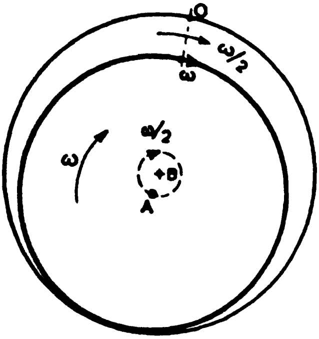

Fig. 123.

For perfect lubrication between two surfaces, one of which is moving, the oil film separating them is *wedge-shaped*† and to cause this the shaft will set itself eccentric in the bearing as indicated in exaggerated form in Fig. 123. One surface of the oil, being in contact with the fixed bearing, is stationary, the other is moving at shaft speed ω, hence the mean velocity of the oil film is about $\frac{\omega}{2}$, and due to the wedging action of the oil the shaft centre A will rotate about the bearing centre B at this speed. Thus when this speed $= \omega_c$, resonant conditions again exist and a circular vibration is built up, but the shaft is not now whirling in a fixed bent position, since its rotation about its own centre is $2\omega_c$, and that of the whirl ω_c. A horizontal shaft is not so likely to exhibit this phenomenon as a vertical, since its weight tends to keep it at the bottom of the bearing and so resist the wedging action of the oil film.

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* " Shaft whipping due to oil action ", B. L. Newkirk, *General Electric Review*, August 1925.

† Beauchamp Towers' Experiments on Lubrication. The Mitchell Thrust bearing depends on this action.

EXAMPLES V

CHAPTERS XII AND XIII

1. The fundamental frequency of transverse vibration of a uniform steel shaft 4 ft. long and ½ in. in diameter was found to be 1060 vibrations per minute. Determine the fundamental frequency of a shaft of the same material and similarly supported, if the span is 6 ft. and the diameter ¾ in. *Ans.* 706·7 per min.

2. A surface condenser has brass tubes ¾ in. external diameter, 0·048 in. thick and 10 ft. long. Given that the equation of the static deflexion curve for a beam fixed at each end is

$$y = \frac{w}{24EI} \cdot x^2(L - x)^2,$$

the origin being taken at one end, and w being the load per unit length, calculate the natural frequency of transverse vibration of a tube when full of water.

Weight of brass, 0·3 lb. per cubic in.
E for brass, 12×10^6 lb. per sq. in.
Weight of water, 62·5 lb. per cubic ft. [Liv. U.]

Ans. 6·42 per sec.

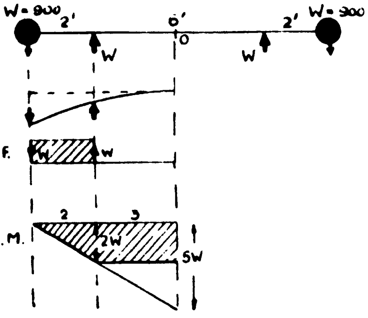

FIG. (*a*).

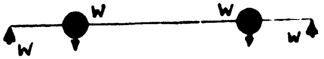

FIG. (*b*).

3. A shaft 10 ft. long and 4 in. in diameter rests on two supports situated 2 ft. from either end. A load of 900 lb. is concentrated at each end of the shaft. Neglecting the mass of the shaft and taking $E = 30 \times 10^6$ lb. per sq. in., calculate the fundamental frequency of transverse vibration. Fig. (*a*).

To find the static deflexion of each load, consider either half of beam, the tangent being horizontal at the mid-point O.

Now the deflexion of any point A from the tangent at another point O, besides being equal to the area of the slope curve as stated above, is also equal to the moment of the $\frac{M}{EI}$ curve between the two points about A.

The bending moment diagram is shown, thus deflexion of the load W from tangent at O

$$\delta_a = \frac{2 \,.\, 900}{EI}\left[3 \,.\, \left(2+\frac{3}{2}\right)+\frac{1}{2} \,.\, 2\left(\frac{2}{3} \,.\, 2\right)\right]$$

$$=\frac{21{,}300}{EI}\text{ ft.}$$

Deflexion of support from tangent at O

$$\delta_b = \frac{2 \,.\, 900}{EI}\left[3 \,.\, \frac{3}{2}\right] = \frac{8{,}100}{EI}\text{ ft.};$$

$$\therefore \text{ Deflexion of } W \text{ below support} = y_1 = \delta_a - \delta_b = \frac{13{,}200}{EI}\text{ ft.}$$

Similarly for the other side.

Whence equating Max. K.E. of system to Max. P.E. we get

$$p^2 = \frac{g}{y_1} = \frac{gEI}{13{,}200};$$

$\therefore$ frequency = 12·62 vibrations per sec.

Alternatively :

Upon examination of the given arrangement it will be clear that if we interchange the loads and supports we shall not alter the static deflexion curve or the frequency of the system. Fig. (*b*).

4. The static deflexion y_s of any point distant x from the fixed end of a uniform cantilever is

$$y_s = \frac{w}{24EI}\left[6L^2x^2 - 4Lx^3 + x^4\right],$$

where L is the length of the cantilever and w is the weight per unit length.

Determine approximately the fundamental frequency of transverse vibration of a uniform steel cantilever, given :

Length - -	3 ft.
Breadth - -	2 in.
Depth - -	½ in.
Density of steel -	0·28 lb. per cu. in.
E - - -	30×10^6 lb. per sq. in.

[Liv. U.]

Ans. 12·7 vibrations per sec.

5. A steel turbine blade, having a uniform cross-section area of 0·124 sq. in. and a length of 10 in., was held firmly clamped at one end, and the fundamental frequency of transverse vibration was found experimentally to be 91·5 per sec. Determine the second moment of area of the cross-section about the neutral axis.

E for steel, 30×10^6 lb. per sq. in.

Weight of steel, 0·28 lb. per cu. in.

Ans. 0·0008 in.^{4}

6. A steel shaft 2 ft. long and $\frac{3}{4}$ in. in diameter is mounted in spherically seated bearings at its ends. Determine the first whirling speed

(*a*) when unloaded,

(*b*) when loaded with 40 lb. at the centre of its length.

$E = 30 \times 10^6$ lb. per sq. in.

Wt. of steel, 0·28 lb. per cu. in. [Liv. U.]

Ans. (*a*) 103·8 per sec.

(*b*) 19·45 per sec.

7. A turbine rotor consists of a shaft of variable cross-section supporting a series of wheels. This system has been considered as nine concentrated loads, which include the weights of the sections of shaft concerned; and the loads and corresponding static deflexions of the shaft at their points of application are given below:

w	lb.	10	43	7	157	150	150	15	25	10
y	in.	26	106	132·5	136	134	122·3	112	84·5	26

(The values of y have been multiplied by 10^5.)

Determine the first whirling speed. [Liv. U.]

Ans. 5250 r.p.m.

8. A wheel of weight 2,000 lb. is mounted at the mid-point of a tapered shaft which is supported freely at its ends.

The taper is such that the second moment of area at every cross-section varies directly as the bending moment at that section.

The length of the shaft is 4 ft. and the diameter at the mid-point is 3 in. Taking $E = 30 \times 10^6$ lb. per sq. in. and neglecting the mass of the shaft, determine the whirling speed. [Liv. U.]

Ans. 780 r.p.m.

(We know that $\frac{M}{I} = \frac{E}{R}$, thus radius of curvature is constant, i.e. the beam bends into the arc of a circle.)

9. A shaft of length 10 ft. and of negligible mass is supported at its ends and carries 5 concentrated loads of 100 lb. each at distances of 1, 2, 5, 8 and 9 ft. respectively from one end. The diameter of the shaft is varied in such a manner that the static deflexion curve for the loaded shaft is approximately a sine curve, the deflexion of any point distant x from one end being given by

$$y_s = 0{\cdot}001 \sin \pi \,.\, \frac{x}{L} \text{ in.},$$

where L = length of shaft.

Determine the fundamental frequency of transverse vibration of the system. [Liv. U.]

Ans. 120 per sec.

10. The centre of gravity of a de Laval rotor does not coincide with the wheel centre. The shaft is run up slowly through its whirling speed and a chalkmark is made on the rim of the disk when at its maximum displacement. State where this mark will appear in relation to the C.G. and explain the reason. [Liv. U.]

11. A continuous beam of negligible mass carries a single load W shown in Fig. (c). Assuming that all the supports are short,

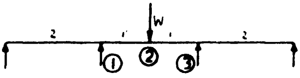

FIG. (c).

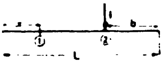

FIG. (d).

determine the natural frequency of transverse vibration (in terms of E, I, and W) given that:

The deflexion produced at a point (1) of a uniform beam, freely supported at its ends, by a unit load at a point (2) is

$$y_{12} = \frac{bx}{6EI \,.\, L}\,[L^2 - b^2 - x^2],$$

the symbols having the meanings shown in Fig. (d). [Liv. U.].

From the general equations of motion:

$$(1) \quad 0 = -R_1 y_{11} + M_2 y_{12} p^2 Y_2 \qquad - R_3 y_{13},$$

$$(2) \quad 0 = -R_1 y_{21} + (M_2 y_{22} p^2 - 1)\, Y_2 - R_3 y_{23},$$

$$(3) \quad 0 = -R_1 y_{31} + M_2 y_{23} p^2 Y_2 \qquad - R_3 y_{33}.$$

Here, from symmetry,

$$R_1 = R_3 = R,$$

$$y_{11} = y_{33},$$

$$y_{21} = y_{23},$$

so that equations (1) and (3) are the same.

From (1) and (2),

$$0 = -R(y_{11} + y_{13}) + M_2 y_{12} p^2 Y_2,$$

$$0 = -R(y_{21} + y_{23}) + (M_2 y_{22} p^2 - 1) Y_2\,;$$

$$\therefore\ \frac{(y_{11} + y_{13})}{(y_{21} + y_{23})} = \frac{M_2 y_{12} p^2}{(M_2 y_{22} p^2 - 1)},$$

$$y_{11} = \frac{32}{9EI} \qquad y_{22} = \frac{9}{2EI}$$

Whence, substituting above,

$$y_{12} = \frac{23}{6EI}\ (= y_{23})$$

$$p^2 = \frac{120}{11} \cdot \frac{EIg}{W},$$

$$y_{13} = \frac{28}{9EI}$$

or frequency $= \dfrac{p}{2\pi}$.

Alternatively, since there is only a single load, the frequency is given by $\dfrac{1}{2\pi}\sqrt{\dfrac{g}{\delta_{st}}}$ where δ_{st} is the static deflexion at the load point, which can be determined as follows:

W produces at (1) a deflexion $Wy_{12} = W \cdot \dfrac{23}{6EI}$,

R_3 produces at (1) a deflexion $-R_3 y_{13}$.

Thus, equation (1) becomes

$$0 = -R_1 y_{11} + W y_{12} - R_3 y_{13},$$

and as $R_1 = R_3 = R$,

$$\therefore\ R = \frac{23}{40} W.$$

Now R_1 produces at (2) a deflexion $-R_1 y_{21} = -\dfrac{23}{40} W \cdot \dfrac{23}{6EI}$.

Similarly R_3 produces at (2) a deflexion $-R_3 y_{23} = -\dfrac{23}{40} W \cdot \dfrac{23}{6EI}$.

$\therefore$ Total deflexion at (2), i.e. $\delta_{st} = W y_{22} - 2\left[\dfrac{W}{EI} \cdot \dfrac{23 \,.\, 23}{40 \,.\, 6}\right]$

$$= \frac{W}{EI} \cdot \frac{11}{120}.$$

12. Fig. (*e*) represents a continuous beam on three supports at the same level and carrying equal loads W at the centres of the spans. Determine the two natural frequencies of transverse vibration assuming that the centre support exerts no constraining action, and given that

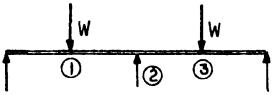

FIG. (*e*).

$$y_{11}=\frac{3}{4EI}, \qquad y_{13}=\frac{7}{12EI},$$

$$y_{12}=\frac{11}{12EI}, \qquad y_{22}=\frac{4}{3EI}.$$

(Note that, due to symmetry, the ratio of the amplitudes of the loads Y_1/Y_3 is $+1$ and -1 respectively in the two modes.)

[Liv. U.]

Ans. $p^2=\frac{EIg}{W}\times\frac{96}{7}$ or 6.

13. A uniform massless beam, supported at its ends, carries three equal loads W lb. symmetrically spaced as shown in Fig. (*f*) i.

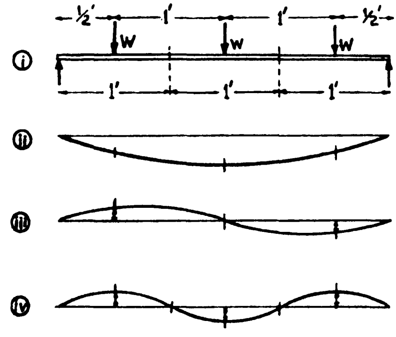

FIG. (*f*).

Write down the general equations of motion of the loads for determining the frequency equation and, making use of the symmetrical shapes taken up in each mode, determine the frequencies of the three modes in terms of W, E, and I, given :

$$y_{12}=\frac{39}{144EI}; \qquad y_{11}=\frac{25}{144EI};$$

$$y_{22}=\frac{81}{144EI}; \qquad y_{13}=\frac{17}{144EI}.$$

[Liv. U.]

The general equations for three loads have been given on page 294. The solution in this case is greatly simplified, since $M_1 = M_2 = M_3$ and

(*a*) for the *fundamental* and *third* modes, $Y_1 = Y_3$, so that equations (1) and (3) are the same. Using equations (1) and (2) a quadratic is formed in p^2,

$$360\left[\frac{Mp^2}{144EI}\right]^2 - 123\left[\frac{Mp^2}{144EI}\right] + 1 = 0,$$

from which $$\frac{Mp^2}{144EI} = \frac{1}{3} \text{ or } \frac{1}{120}. \qquad \text{Fig. } (f) \text{ ii, iv.}$$

Alternatively for the *third* mode, $Y_1 = Y_3 = -Y_2$, so that direct from equation (2),

$$2My_{21}p^2 = (My_{22}p^2 - 1),$$

i.e. $$\frac{Mp^2}{144EI} = \frac{1}{3}. \qquad \text{Fig. } (f) \text{ iv.}$$

Again for this mode, taking a single bow (length 1 ft.), this is equivalent to a beam freely supported at its ends and carrying a concentrated load W at the centre. The static deflexion δ_{st} for this case is $\frac{WL^3}{48EI}$ and is probably known to the student (or can be obtained by the methods previously employed), so that

$$p^2 = \frac{g}{\delta_{st}} = \frac{g \cdot 48EI}{W \cdot 1},$$

or $$\frac{Mp^2}{144EI} = \frac{1}{3}.$$

For the *second* mode a node is formed at W_2, thus

$$Y_1 = -Y_3,$$

$$Y_2 = 0, \qquad \text{Fig. } (f) \text{ iii.}$$

and substituting in (say) equation (1) we get

$$\frac{Mp^2}{144EI} = \frac{1}{8}.$$

Alternatively, taking half the total span (length 1½ ft.), this is equivalent to a beam freely supported at its ends and carrying the single load W, the static deflexion of which can be determined from our previous methods.

14. A uniform shaft is supported in four bearings which divide it into three equal spans. A concentrated load W lb. is carried at the centre of each of the two outer spans. Assuming that all the bearings are short so that the influence numbers can be determined from the formula on page 295, determine the frequencies of the two natural modes of transverse vibration in terms of E, I and W.

Ans. $p^2 = \dfrac{240}{31} \cdot g \dfrac{EI}{W}$

$\dfrac{48}{5} \cdot g \dfrac{EI}{W}$

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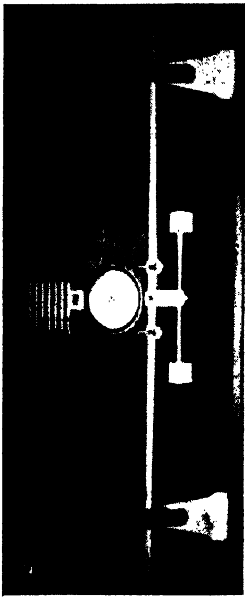
Torsional Vibration in Diesel Engines. C. M. Bradbury (C. Griffin). A most concise little book on this section of the subject.

More advanced works :

Mechanical Vibrations. Den Hartog (McGraw-Hill). A most readable book which every student should possess.

Practical Solution of Torsional Vibration Problems. W. Ker Wilson (Chapman and Hall). A standard work in two volumes.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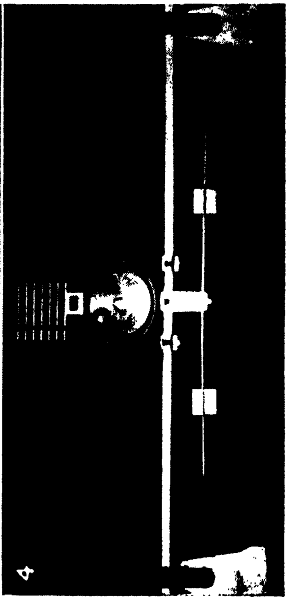
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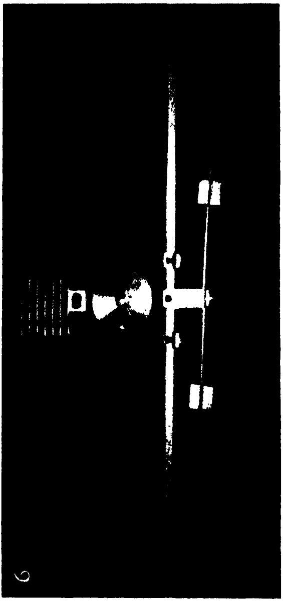
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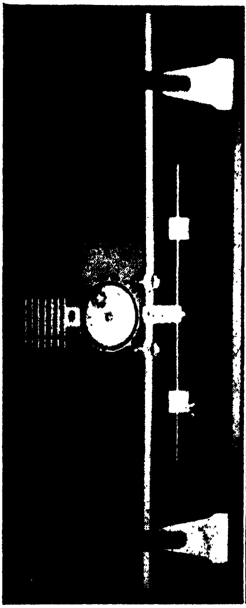

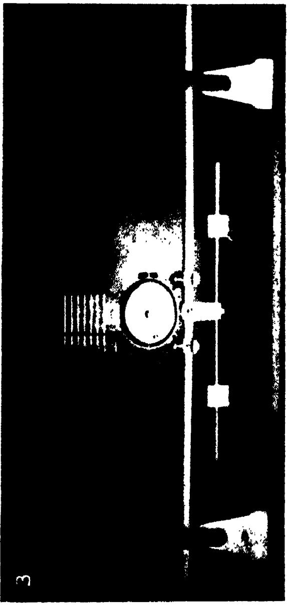
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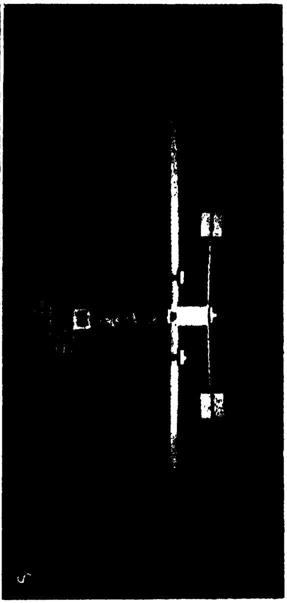
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APPENDIX A

THE UNDAMPED VIBRATION ABSORBER

FIG. A. 1 shows a very effective demonstration absorber. An electric motor, fixed at the centre of the span of a horizontal steel beam, drives a disk carrying an out-of-balance mass, which causes the system to be set into severe vibration in a vertical plane. The extent of the vibration can be judged by comparison of the horizontal white lines on the scale attached to the motor with the corresponding scale at the side which is not influenced by the vibration. The motor runs at constant speed. The absorber is fixed below the beam and consists of two equal masses (for symmetry) which can be clamped in any positions along two small horizontal steel cantilevers. In their innermost positions these masses would form with the motor the equivalent of a single concentrated load on the main beam.

Fig. A. 2 : The absorber masses are incorrectly adjusted and form with the motor a 3-mass system which is seen to be in severe vibration. The exposure was made over a series of cycles, so that the vibration of the main system is indicated by the blurring of the white line scale on the motor.

Fig. A. 3 : The absorber is tuned correctly and is taking the whole energy of vibration (with very moderate amplitude) while the motor and beam are free from vibration.

Fig. A. 4 : An instantaneous photograph of condition A. 3, showing that the two absorber masses both move in the same direction together.

The last two photographs, showing severe 3-mass vibration, are inserted as a matter of interest.

Fig. A. 5 : An instantaneous photograph of condition A. 2 showing that with this setting the absorber masses and the motor move in the same direction together.

Fig. A. 6 : Here the adjustment is such that the absorber masses move in opposite directions.

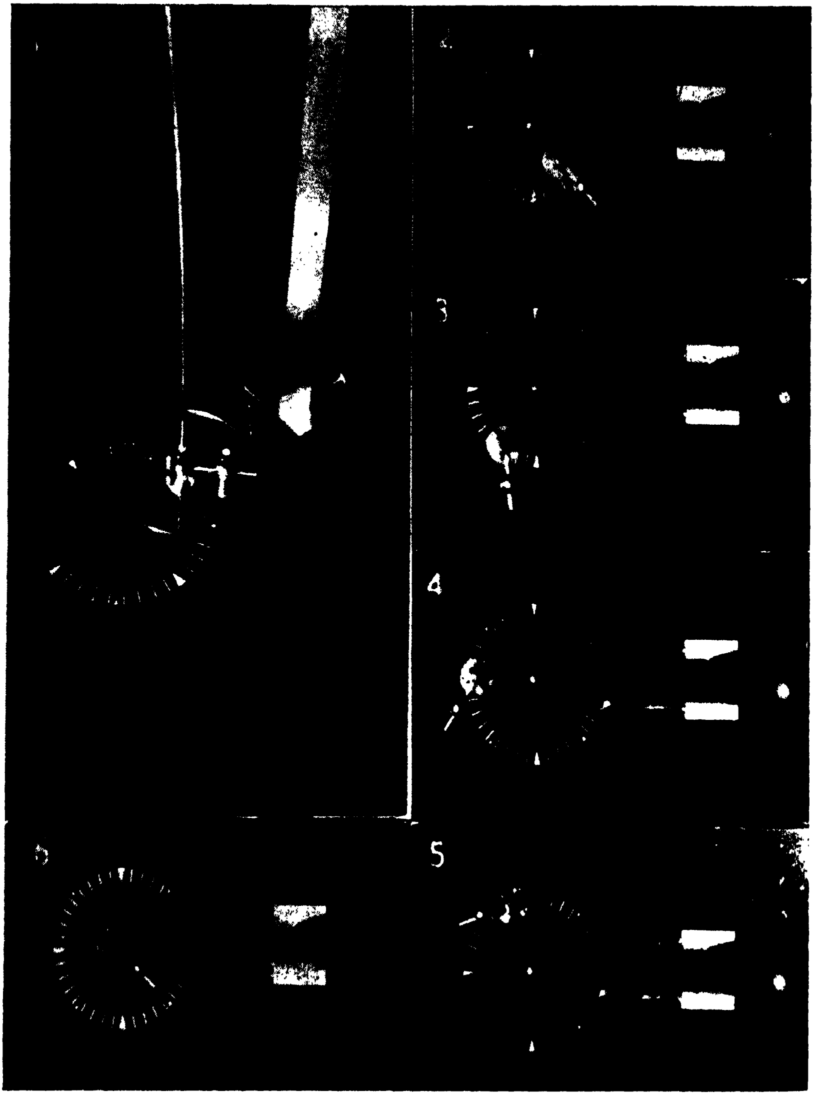

Fig. B.

APPENDIX B

THE WHIRLING OF A DE LAVAL ROTOR

FIG. B. 1 shows a transparent disk mounted at the centre of the span of a light vertical shaft. The centre of gravity of the disk has been made eccentric by the attachment of an out-of-balance mass and will lie somewhere on the white radial line through this mass.

The remaining photographs were taken along the axis of the shaft and, the top bearing being out of focus, the centre line is indicated by a white spot, being fixed from the graduations on the guard ring.

The photographs, although apparently instantaneous, were actually " time ", since the motion was apparently stopped by illuminating the system stroboscopically, a contact on the motor driving shaft causing a flash from the stroboscopic lamp *at exactly the same point in each revolution.*

The rotation of the disk round its own centre and the rotation of the circular whirl of the shaft was anti-clockwise.

Figs. B. 2 → 5 show how the plane of bending of the shaft changed relatively to the radial white line as the speed was brought up towards the critical.

Fig. B. 6 shows conditions above the critical, the C.G. now lying in the plane of bending again but within the bow.

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